# On quasicoherent sheaves on toric schemes

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# Aim

Study toric schemes

$$X_\Sigma(R) \longrightarrow \mathrm{Spec}(R)$$

over arbitrary affine bases

$\rightsquigarrow$ study quasicoherent sheaves on $X_\Sigma(R)$

# Analogy

Compare with projective schemes

$$\mathrm{Proj}(S) \longrightarrow \mathrm{Spec}(R)$$

over arbitrary affine bases

$\rightsquigarrow$ compare with quasicoherent sheaves on $\mathrm{Proj}(S)$

# Overview

1. Reminder on projective schemes
2. Quasicoherent sheaves on toric schemes
3. Finiteness conditions

# Projective schemes

- $R$: arbitrary base ring
- $S$: positively $\mathbb{Z}$-graded $R$-algebra, generated by finitely many elements of degree 1

$$X = \mathrm{Proj}(S) \to \mathrm{Spec}(R)$$

# Homogeneous coordinate rings I

- coordinate ring $S = \bigoplus_{n \in \mathbb{N}} S_n$
  - positively $\mathbb{Z}$-graded $R$-algebra of finite type
- irrelevant ideal $S_+$
  - graded ideal, generated by finitely many elements of $S_1$

# Homogeneous coordinate rings II

- canonical affine open covering $X = \bigcup_{f \in S_1} \mathrm{Spec}(S_{(f)})$
- has non-canonical finite subcovering

# Quasicoherent sheaves on projective schemes I

## Theorem

a) *There is an essentially surjective, exact functor*

$$\widetilde{\bullet} : \mathrm{GrMod}^{\mathbb{Z}}(S) \to \mathrm{QCMod}(\mathscr{O}_X)$$

*that vanishes precisely on $S_+$-torsion modules*

# Quasicoherent sheaves on projective schemes II

## Theorem (continued)

b) $\widetilde{\bullet}$ *induces for every* $\mathbb{Z}$*-graded* $S$*-module* $F$ *a bijection*

$$\left\{ \begin{array}{c} S_{+}\text{-saturated graded} \\ \text{sub-}S\text{-modules of } F \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{quasicoherent sub-} \\ \mathscr{O}_X\text{-modules of } \widetilde{F} \end{array} \right\}$$

# Toric schemes

- $R$: arbitrary base ring
- $V$: $\mathbb{R}$-vector space of finite dimension
- $N$: $\mathbb{Z}$-structure on $V$
- $\Sigma$: full $N$-fan in $V$

$$X = X_{\Sigma}(R) \to \operatorname{Spec}(R)$$

# Homogeneous coordinate rings I

- coordinate ring $S = \bigoplus_{\alpha \in B} S_\alpha$
  - $B$-graded sub-$R$-algebra of polynomial $R$-algebra, generated by monomials
  - $B$: finitely generated commutative group, e.g.
    - cokernel of canonical morphism $N^* \to \mathbb{Z}^{\Sigma_1}$
    - (if $\Sigma$ simplicial) Picard group $P = \mathrm{Pic}(\Sigma)$ of $\Sigma$

# Homogeneous coordinate rings II

- irrelevant ideal $I$
  - graded ideal, generated by finitely many monomials
- canonical finite affine open covering $X = \bigcup_{\sigma \in \Sigma} \mathrm{Spec}(S_{(\sigma)})$
  - $S_{(\sigma)}$: component of degree 0 of ring of fractions obtained by inverting certain monomial

# Simplicial case I

## Theorem

a) *There is an essentially surjective, exact functor*

$$\widetilde{\bullet} : \mathsf{GrMod}^P(S) \to \mathsf{QCMod}(\mathscr{O}_X)$$

*that vanishes precisely on $I$-torsion modules*

Cox (1995): in case $R = \mathbb{C}$

# Simplicial case II

## Theorem (continued)

b) $\widetilde{\bullet}$ *induces for every $P$-graded $S$-module $F$ a bijection*

$$\left\{ \begin{array}{c} I\text{-saturated graded} \\ \text{sub-}S\text{-modules of } F \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{quasicoherent sub-} \\ \mathscr{O}_X\text{-modules of } \widetilde{F} \end{array} \right\}$$

Cox (1995): in case $R = \mathbb{C}$, $F = S$

# General case I

## Theorem

a) *There is an essentially surjective, exact functor*

$$\widetilde{\bullet} : \mathsf{GrMod}^B(S) \to \mathsf{QCMod}(\mathscr{O}_X)$$

*that vanishes on $I$-torsion modules*

Mustață (2002): in case $R = \mathbb{C}$

# General case II

## Theorem (continued)

b) $\widetilde{\bullet}$ *induces for every* $B$*-graded* $S$*-module* $F$ *a surjection*

$$\left\{ \begin{array}{c} I\text{-saturated graded} \\ \text{sub-}S\text{-modules of } F \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{quasicoherent sub-} \\ \mathscr{O}_X\text{-modules of } \widetilde{F} \end{array} \right\}$$

# Some details I

- construction and exactness of $\widetilde{\bullet}$

$$\begin{array}{ccc} \mathrm{GrMod}^B(S) & \xrightarrow{\widetilde{\bullet}} & \mathrm{QCMod}(\mathscr{O}_X) \\ \downarrow{\scriptstyle \bullet_{(\sigma)}} & & \downarrow{\scriptstyle \bullet\restriction_{\mathrm{Spec}(S_{(\sigma)})}} \\ \mathrm{Mod}(S_{(\sigma)}) & \xrightarrow[\text{canon. equiv.}]{\widetilde{\bullet}} & \mathrm{QCMod}(\mathscr{O}_{\mathrm{Spec}(S_{(\sigma)})}) \end{array}$$

# Some details II

- total functor of sections

$$\Gamma_*(\bullet) : \mathrm{QCMod}(\mathscr{O}_X) \to \mathrm{GrMod}^B(S),$$

$$\Gamma_*(\mathscr{F}) = \bigoplus_{\alpha \in B} (\widetilde{S(\alpha)} \otimes_{\mathscr{O}_X} \mathscr{F}(X))$$

# Some details III

- canonical isomorphism $\beta : \widetilde{\Gamma_*(\bullet)} \to \mathrm{Id}_{\mathsf{QCMod}(\mathscr{O}_X)}$

  $\rightsquigarrow$ $\Gamma_*$ is right quasiinverse of $\widetilde{\bullet}$

  $\rightsquigarrow$ $\widetilde{\bullet}$ is essentially surjective

# Some details IV

- a "second total functor of sections"

$$\Gamma_{**}(\bullet) : \mathsf{GrMod}^B(S) \to \mathsf{GrMod}^B(S),$$

$$\Gamma_{**}(F) = \bigoplus_{\alpha \in B} (\widetilde{F(\alpha)}(X))$$

# Some details V

- compare

$$\Gamma_*(\widetilde{F}) = \bigoplus_{\alpha \in B} (\widetilde{S(\alpha)} \otimes_{\mathscr{O}_X} \widetilde{F}(X))$$

and

$$\Gamma_{**}(F) = \bigoplus_{\alpha \in B} (\widetilde{S(\alpha) \otimes_S F(X)})$$

- coincide in the projective case

# Some details VI

- canonical morphism $\delta : \Gamma_*(\widetilde{\bullet}) \to \Gamma_{**}(\bullet)$
  - not necessarily an isomorphism
- canonical morphism $\eta : \mathrm{Id}_{\mathsf{GrMod}^B(S)}(\bullet) \to \Gamma_{**}(\bullet)$

# Some details VII

## Key observation

- *commutative diagram*

$$\begin{array}{ccc} & \widetilde{\Gamma_*(\widetilde{\bullet})} & \\ {}^{\beta(\widetilde{\bullet})}\swarrow & & \searrow^{\widetilde{\delta(\bullet)}} \\ \widetilde{\bullet} & \xrightarrow[\widetilde{\eta(\bullet)}]{} & \widetilde{\Gamma_{**}(\bullet)} \end{array}$$

- *occuring morphisms are isomorphisms*

# Finiteness of homogeneous coordinate rings

**Proposition**

$\mathrm{cone}(\Sigma) = \langle \Sigma \rangle \Rightarrow$ *$S$ positively $B$-graded $R$-algebra of finite type*

- positively $B$-graded: $S_\alpha \neq 0$, $S_{-\alpha} \neq 0 \Rightarrow \alpha = 0$
- converse is not true

# Finite type and finite presentation I

## Proposition

*$F$: $B$-graded $S$-module*
*$\mathscr{G} \hookrightarrow \widetilde{F}$: quasicoherent sub-$\mathscr{O}_X$-module of finite type*
*$\Rightarrow \exists$ graded sub-$S$-module of finite type $G \hookrightarrow F$:*

$$(\widetilde{G \hookrightarrow F}) = (\mathscr{G} \hookrightarrow \widetilde{F})$$

- Reason: $X$ quasicompact, $\widetilde{\bullet}$ commutes with inductive limits

# Finite type and finite presentation II

## Proposition

*If $\Sigma$ is simplicial then $\widetilde{\bullet}$ preserves the properties of being*

- *of finite type*
- *of finite presentation*

- Reason: $S_\sigma$ is strongly $P$-graded $\rightsquigarrow S(\alpha)_{(\sigma)} \cong S_{(\sigma)}$ for $\alpha \in P$

# Pseudocoherence and coherence I

## Proposition

*If $\Sigma$ is simplicial then $\widetilde{\bullet}$ preserves the properties of being*

- *pseudocoherent*
- *coherent*

- pseudocoherent: subobjects of finite type are of finite presentation
- coherent: pseudocoherent and of finite type

# Pseudocoherence and coherence II

## Corollary

*$\Sigma$ N-regular, R stably coherent $\Rightarrow \mathscr{O}_X$ coherent*

- $R$ stably coherent: polynomial algebras in finitely many indeterminates over $R$ are coherent
- coherent $\not\Rightarrow$ stably coherent
- Examples of stably coherent rings: absolutely flat rings, valuation rings, semihereditary rings...