On quasicoherent sheaves on toric schemes

Fred Rohrer
Viện Toán học, Hà Nội

Hội nghị Đại số – Hình học – Tôpô
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Projective and toric schemes

We consider projective schemes and toric schemes

$$\begin{array}{ccc} \mathrm{Proj}(S) & \qquad & X_\Sigma(R) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & & \mathrm{Spec}(R) \end{array}$$

over arbitrary affine bases.

$\rightsquigarrow$ very strong analogy between quasicoherent sheaves on schemes of these two classes

The projective setting

- R: arbitrary base ring
- S: positively $\mathbb{Z}$-graded R-algebra, generated by finitely many elements of degree 1

$$X = \mathrm{Proj}(S) \to \mathrm{Spec}(R)$$

The toric setting

- R: arbitrary base ring
- V: $\mathbb{R}$-vector space of finite dimension
- N: $\mathbb{Z}$-structure on V
- Σ: full simplicial N-fan in V

$$X = X_\Sigma(R) \to \mathrm{Spec}(R)$$

Homogeneous coordinate rings

proj.

- coordinate ring $S = \bigoplus_{n \in \mathbb{N}} S_n$
- irrelevant ideal S_+
- $X = \bigcup_{f \in S_1} \mathrm{Spec}(S_{(f)})$

toric

- coordinate ring $S = \bigoplus_{\alpha \in P} S_\alpha$ with P the Picard group of Σ
- irrelevant ideal I
- $X = \bigcup_{\sigma \in \Sigma} \mathrm{Spec}(S_{(\sigma)})$

Quasicoherent sheaves associated with graded modules

The projective case

Theorem (classical)

- *There is an essentially surjective, exact functor*

$$\widetilde{\bullet} : \mathrm{GrMod}^{\mathbb{Z}}(S) \to \mathrm{QCMod}(\mathscr{O}_X)$$

- $\widetilde{\bullet}$ *induces for every $\mathbb{Z}$-graded S-module F a bijection*

$$\left\{ \begin{array}{c} S_+\text{-saturated graded} \\ \text{sub-}S\text{-modules of } F \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{quasicoherent sub-} \\ \mathscr{O}_X\text{-modules of } \widetilde{F} \end{array} \right\}$$

- $\widetilde{F} = 0$ *if and only if F is S_+-torsion*

Quasicoherent sheaves associated with graded modules

The toric case

Theorem (Cox (1995) for $R = \mathbb{C}$, $F = S$, - (2011) in general)

- *There is an essentially surjective, exact functor*

$$\widetilde{\bullet} : \mathrm{GrMod}^P(S) \to \mathrm{QCMod}(\mathscr{O}_X)$$

- $\widetilde{\bullet}$ *induces for every P-graded S-module F a bijection*

$$\left\{ \begin{array}{c} I\text{-saturated graded} \\ \text{sub-}S\text{-modules of } F \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{quasicoherent sub-} \\ \mathscr{O}_X\text{-modules of } \widetilde{F} \end{array} \right\}$$

- $\widetilde{F} = 0$ *if and only if F is I-torsion*

Finiteness conditions
The toric case

Theorem (- (2011))

- Σ *skeletal complete* $\Rightarrow$ *S positively P-graded R-algebra of finite type*
- $\widetilde{\bullet}$ *preserves the properties of being of finite type, of finite presentation, pseudocoherent, and coherent*
- Σ *N-regular, R stably coherent* $\Rightarrow$ $\mathscr{O}_X$ *coherent*