AROUND
“AROUND CASTELNUOVO-MUMFORD REGULARITY”

FRED ROHRER

Dedicated to Markus Brodmann on the occasion of his retirement

These notes are a collection of various results from commutative and homological algebra, mostly concerned with local cohomology. They were motivated by Markus Brodmann's lecture *“Around Castelnuovo-Mumford regularity”* at the University of Zurich during spring semester 2010, and presented and discussed during the corresponding exercise sessions. Throughout, the aim is the generalisation of some notions and results from the case of homogeneous $\mathbb{Z}$-graded algebras over a field (corresponding to classical projective geometry) to arbitrarily graded and not necessarily Noetherian rings (corresponding to toric geometry). For unexplained notation and terminology the reader is referred to [13].

Throughout the following, let $\psi : G \twoheadrightarrow H$ and $\varphi : F \rightarrowtail G$ be an epi- and a monomorphism in Ab*, and let R be a G-graded ring.*

1. Quasidivisibility

We study divisibility, and more general quasidivisibility, of modules, and investigate the property of a ring that injective modules are quasidivisible, called the IQD-property. Our main result identifies some classes of rings having the IQD-property.

Throughout this section let $M \in \mathrm{Ob}(\mathsf{GrMod}^G(R))$.

(1.1) A) We define a map $\delta_M : M^{\mathrm{hom}} \to G$ by setting $\delta_M(s) := \deg(s)$ for $s \in M^{\mathrm{hom}} \setminus 0$ and $\delta_R(0) := 0$. If no confusion can arise then we write δ instead of δ_M. Clearly, it holds $\delta_{M_{[\psi]}} = \psi \circ \delta_M$.

B) If $s \in R^{\mathrm{hom}}$ then the morphism $M \to M(\delta(s))$ in $\mathsf{GrMod}^G(R)$ induced by multiplication with s is denoted by s_M. Clearly, it holds $(s_M)_{[\psi]} = s_{M_{[\psi]}}$.

C) If $s \in R^{\mathrm{hom}}$ then the multiplication morphism s_M has a retraction if and only if it is an isomorphism. Indeed, if $g : M(\delta(s)) \to M$ is a retraction of s_M then it holds $sg(x) = g(sx) = g(s_M(x)) = x$ for every $x \in M$, and hence s_M is a section of g.

(1.2) A) Let $S \subseteq R^{\mathrm{hom}}$. The G-graded R-module M is called *S-divisible* if the multiplication morphism s_M is an epimorphism for every $s \in S$, and *S-quasidivisible* if the multiplication morphism $s_{M/{}^G\Gamma_{\langle s\rangle}(M)}$ is an epimorphism for every $s \in S$. In case $S = \{s\}$ we say s-divisible and s-quasidivisible. Obviously, M is S-quasidivisible if and only if $M/{}^G\Gamma_{\langle s\rangle}(M)$ is s-divisible for every $s \in S$.

B) As 0 is obviously R^{hom}-divisible and we have ${}^G\Gamma_{\langle s\rangle} = \mathrm{Id}_{\mathsf{GrMod}^G(R)}$ for every nilpotent $s \in R$, we see that M is s-quasidivisible for every nilpotent $s \in R$.

Date: Zürich, June 3, 2010.

C) If M is S-divisible then it is S-quasidivisible. Indeed, let $s \in R^{\mathrm{hom}}$ and suppose that s_M is an epimorphism. Composing s_M with the canonical epimorphism

$$M(\delta(s)) \twoheadrightarrow M(\delta(s))/{}^G\Gamma_{\langle s\rangle}(M(\delta(s))) = (M/{}^G\Gamma_{\langle s\rangle}(M))(\delta(s))$$

yields an epimorphism that equals the composition of the canonical epimorphism $M \twoheadrightarrow M/{}^G\Gamma_{\langle s\rangle}(M)$ with $s_{M/{}^G\Gamma_{\langle s\rangle}(M)}$. Therefore, $s_{M/{}^G\Gamma_{\langle s\rangle}(M)}$ is an epimorphism, and hence M is s-quasidivisible.

(1.3) Proposition *If $S \subseteq R^{\mathrm{hom}}$ then M is S-(quasi-)divisible if and only if $M_{[\psi]}$ is S-(quasi-)divisible.*

Proof. Let $s \in S$. Then, s_M is an epimorphism if and only if $s_{M_{[\psi]}} = (s_M)_{[\psi]}$ is an epimorphism (see [13, III.2.1.3]), and so the claim about divisibility is proven. Moreover, M is s-quasidivisible if and only if $M/{}^G\Gamma_{\langle s\rangle}(M)$ is s-divisible, and by the above this holds if and only if $M_{[\psi]}/{}^H\Gamma_{\langle s\rangle_{[\psi]}}(M_{[\psi]}) = (M/{}^G\Gamma_{\langle s\rangle}(M))_{[\psi]}$ is s-divisible (see [13, III.2.1.3; 4.4.4]). But this is equivalent to $M_{[\psi]}$ being s-quasidivisible, and so the claim about quasidivisibility is proven. □

(1.4) A) We set $\mathrm{NZD}_R(M) := \{x \in R^{\mathrm{hom}} \mid \mathrm{Ker}(x_M) = 0\}$ and $\mathrm{ZD}_R(M) := R^{\mathrm{hom}} \setminus \mathrm{NZD}_R(M)$, and we set $\mathrm{NZD}(R) := \mathrm{NZD}_R(R)$ and $\mathrm{ZD}(R) := \mathrm{ZD}_R(R)$. So, for $x \in R^{\mathrm{hom}}$ it holds $x \in \mathrm{ZD}_R(M)$ if and only if there exists $y \in M^{\mathrm{hom}} \setminus 0$ with $xy = 0$.

B) Clearly, we have $\mathrm{ZD}_R(M) = \mathrm{ZD}_{R_{[\psi]}}(M_{[\psi]}) \cap R^{\mathrm{hom}} \subseteq \mathrm{ZD}_{R_{[\psi]}}(M_{[\psi]})$ and $\mathrm{NZD}_R(M) = \mathrm{NZD}_{R_{[\psi]}}(M_{[\psi]}) \cap R^{\mathrm{hom}} \subseteq \mathrm{NZD}_{R_{[\psi]}}(M_{[\psi]})$.

C) If $\mathfrak{a} \subseteq R$ is a G-graded ideal with $\mathfrak{a} \cap \mathrm{NZD}_R(M) \neq \emptyset$ then it clearly holds ${}^G\Gamma_{\mathfrak{a}}(M) = 0$.

(1.5) Proposition *If $s \in S$ then the following statements are equivalent:*

(i) M is s-quasidivisible;

(ii) The multiplication morphism $s_{M/{}^G\Gamma_{\langle s\rangle}(M)}$ is an isomorphism;

(iii) The canonical morphism $\eta_s : M \to M_s$ is an epimorphism;

(iv) It holds ${}^G H^1_{\langle s\rangle}(M/{}^G\Gamma_{\langle s\rangle}(M)) = 0$.

Proof. Since $s \in \mathrm{NZD}_R(M/\Gamma_{\langle s\rangle}(M))$ it is clear that (i) and (ii) are equivalent. So, suppose that $s_{M/\Gamma_{\langle s\rangle}(M)}$ is an isomorphism, and let $x \in M_s$. Then, there are $y \in M$ and $n \in \mathbb{N}_0$ with $x = \frac{y}{s^n}$, and hence there are $z \in M$ and $m \in \mathbb{N}_0$ with $s^m(y - s^n z) = 0$, implying $x = \frac{y}{s^n} = \frac{s^n z}{s^n} = \eta_s(z)$. This shows that (ii) implies (iii). Conversely, suppose that $\eta_s : M \to M_s$ is an epimorphism. Since $\mathrm{Ker}(\eta_s) = \Gamma_{\langle s\rangle}(M)$ it follows $M/\Gamma_{\langle s\rangle}(M) \cong M_s$, and as s_{M_s} is an isomorphism the same holds for $s_{M/\Gamma_{\langle s\rangle}(M)}$. Therefore, (iii) implies (ii).

Finally, we show that (iii) and (iv) are equivalent. Applying local cohomology to the exact sequence

$$0 \to M/{}^G\Gamma_{\langle s\rangle}(M) \to M_s \to \mathrm{Coker}(\eta_s) \to 0$$

yields an exact sequence

$${}^G\Gamma_{\langle s\rangle}(M_s) \to {}^G\Gamma_{\langle s\rangle}(\mathrm{Coker}(\eta_s)) \to {}^G H^1_{\langle s\rangle}(M/{}^G\Gamma_{\langle s\rangle}(M)) \to {}^G H^1_{\langle s\rangle}(M_s).$$

As s_{M_s} is an isomorphism the same holds for $s_{{}^G\Gamma_{\langle s\rangle}(M)}$ and $s_{{}^G H^1_{\langle s\rangle}(M)}$. Moreover, as ${}^G\Gamma_{\langle s\rangle} \circ {}^G\Gamma_{\langle s\rangle} = {}^G\Gamma_{\langle s\rangle}$ and ${}^G\Gamma_{\langle s\rangle} \circ {}^G H^1_{\langle s\rangle} = {}^G H^1_{\langle s\rangle}$ by [13, III.4.4.9] we get

${}^{G}H^{1}_{\langle s\rangle}(M/{}^{G}\Gamma_{\langle s\rangle}(M)) \cong {}^{G}\Gamma_{\langle s\rangle}(\mathrm{Coker}(\eta_s))$. Since we obviously have $\mathrm{Coker}(\eta_s) = {}^{G}\Gamma_{\langle s\rangle}(\mathrm{Coker}(\eta_s))$ the claim is proven. □

(1.6) If $S \subseteq R^{\mathrm{hom}}$ then we say that R has *the IQD-property with respect to S* if every injective G-graded R-module is S-quasidivisible. In case $S = \{s\}$ we speak of the IQD-property with respect to s, and in case $S = R^{\mathrm{hom}}$ we just speak of the IQD-property.

(1.7) Proposition *Let M be a G-graded R-module. If M is injective then it is $\mathrm{NZD}(R) \cup \mathrm{NZD}_R(M)$-divisible.*

Proof. First, let $s \in \mathrm{NZD}(R)$, let $x \in M^{\mathrm{hom}}$, and let $g := -\delta(x)$. There is a unique morphism $f : R(g) \to M$ in $\mathsf{GrMod}^G(R)$ with $f(1) = x$ (see [13, III.2.3.3]). Since the multiplication morphism $s_{R(g)}$ is a monomorphism, the induced map

$$\mathrm{Hom}_{\mathsf{GrMod}^G(R)}(R(g+\delta(s)), M) \to \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(R(g), M),\ f \mapsto f \circ s_{R(g)}$$

is surjective. So, there is a morphism $g : R(g+\delta(s)) \to M$ with $f = g \circ s_{R(g)}$, hence with $x = f(1) = g(s_{R(g)}(1)) = sg(1)$. This shows that s_M is an epimorphism, and so M is s-divisible.

Next, let $s \in \mathrm{NZD}_R(M)$. Then, the multiplication morphism s_M is a monomorphism, hence has a retraction g and thus is an isomorphism by 1.1 C). Therefore, M is s-divisible. □

(1.8) A) The G-graded ring R is called *hereditary* if every monomorphism in $\mathsf{GrMod}^G(R)$ with a projective target has a projective source, that is, G-graded sub-R-modules of projective G-graded R-modules are projective. On use of [13, III.2.1.1] it can be shown by general nonsense that R is hereditary if and only if every epimorphism in $\mathsf{GrMod}^G(R)$ with injective source has an injective target, that is, G-graded quotients of injective G-graded R-modules are injective (see [14, Theorem 4.23]).

B) If $R_{[\psi]}$ is hereditary then R is hereditary, too, as is seen on use of the fact that projectivity and monomorphisms are respected and reflected by coarsenings (see [13, III.2.1.3; 2.4.6]).

(1.9) The G-graded ring R is called *integral* if $\mathrm{ZD}(R) = \{0\}$, that is, if for all $x, y \in R^{\mathrm{hom}} \setminus 0$ it holds $xy \neq 0$, and moreover $R \neq 0$. It is easy to see that if $R_{[\psi]}$ is integral then R is integral, too. The converse holds if G is torsionfree, as is seen on use of [5, Satz 17.25].

(1.10) Proposition *Let $S \subseteq R^{\mathrm{hom}}$ and suppose that R fulfils one of the following conditions:*

1) R is hereditary;

2) Every element of $\mathrm{ZD}(R)$ is nilpotent;

3) R is integral;

4) R has the ITR-property with respect to $\langle s\rangle$ for every $s \in S$;

5) R has the ITI-property with respect to $\langle s\rangle$ for every $s \in S$;

6) R is Noetherian.

Then, R has the IQD-property with respect to S.

Proof. Let I be an injective G-graded R-module. If R is hereditary and $s \in S$ then $I/\Gamma_{\langle s\rangle}(I)$ is injective by 1.8 A), hence s-divisible by 1.7, and thus I is s-quasidivisible.

Next, suppose that every element of $\mathrm{ZD}(R)$ is nilpotent, and let N denote the set of nilpotent elements of R^{hom}. We know from 1.7 that I is $\mathrm{NZD}(R)$-divisible, and moreover it is N-quasidivisible. So, it follows from 1.2 C) that it is $\mathrm{NZD}(R) \cup N$-quasidivisible. But the hypothesis implies $R^{\mathrm{hom}} = \mathrm{NZD}(R) \cup \mathrm{ZD}(R) = \mathrm{NZD}(R) \cup N$, and hence I is R^{hom}-quasidivisible, thus S-quasidivisible.

If R is integral then every element of $\mathrm{ZD}(R) = \{0\}$ is nilpotent, and hence condition 2) is fulfilled.

Now, let $s \in S$ and suppose that R has the ITR-property with respect to $\langle s\rangle$. It clearly holds ${}^{G}H^{1}_{\langle s\rangle}(I) = 0$, hence ${}^{G}H^{1}_{\langle s\rangle}(I/\Gamma_{\langle s\rangle}(I)) \cong {}^{G}H^{1}_{\langle s\rangle}(I) = 0$ by [13, III.4.4.10], and thus I is s-quasidivisible by 1.5.

Finally, if R is Noetherian then it has the ITI-property by [13, III.3.6.6], and if $s \in S$ and R has the ITI-property with respect to $\langle s\rangle$ then it has the ITR-property with respect to $\langle s\rangle$ by [13, III.4.2.3], so in both cases the claim follows from the above. □

(1.11) By [13, III.3.6.6; 4.2.3] and 1.10 we have the implications "R is Noetherian $\Rightarrow$ R has the ITI-property $\Rightarrow$ R has the ITR-property $\Rightarrow$ R has the IQD-property", and we ask if there are further implications between these conditions. As there are integral rings that are non-Noetherian (for example polynomial rings in infinitely many indeterminates over integral rings) it follows from 1.10 that a ring with the IQD-property is not necessarily Noetherian, and hence the above conditions are not equivalent. It would be interesting to know if there exists a ring without the IQD-property.

2. The comparison sequence of local cohomology

On use of the triad sequence and under some conditions on the base ring (more precisely, on the behaviour of injective modules) we construct the comparison sequence of local cohomology; this is heavily based on [4, 1.4]. As an application we prove a graded version of Hartshorne's Vanishing Theorem.

Throughout this section let $\mathfrak{a} \subseteq R$ be a G-graded ideal, and let $b \in R^{\mathrm{hom}}$.

(2.1) A) In $\mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$ we have a canonical exact sequence

$$0 \longrightarrow {}^{G}\Gamma_{\langle b\rangle} \longrightarrow \mathrm{Id}_{\mathsf{GrMod}^G(R)} \longrightarrow \bullet_b,$$

and composing this with ${}^{G}\Gamma_{\mathfrak{a}}$ yields an exact sequence

$$0 \longrightarrow {}^{G}\Gamma_{\mathfrak{a}+\langle b\rangle} \xrightarrow{\iota^{G}_{\mathfrak{a},b}} {}^{G}\Gamma_{\mathfrak{a}} \xrightarrow{\eta^{G}_{\mathfrak{a},b}} {}^{G}\Gamma_{\mathfrak{a}}(\bullet)_b.$$

If no confusion can arise then we write $\iota := \iota^{G}_{\mathfrak{a},b}$ and $\eta := \eta^{G}_{\mathfrak{a},b}$. Obviously, it holds $(\iota^{G}_{\mathfrak{a},b})_{[\psi]} = \iota^{H}_{\mathfrak{a}_{[\psi]},b}$ and $(\eta^{G}_{\mathfrak{a},b})_{[\psi]} = \eta^{H}_{\mathfrak{a}_{[\psi]},b}$.

B) Clearly, $(\iota^{G}_{\mathfrak{a},b}, \eta^{G}_{\mathfrak{a},b})$ is a triad if and only if ${}^{G}\Gamma_{\mathfrak{a}}(I)$ is b-quasidivisible for every injective G-graded R-module I.

(2.2) Proposition *If R has the ITI-property with respect to $\mathfrak{a}$ and the IQD-property with respect to $\langle b\rangle$, then $(\iota^{G}_{\mathfrak{a},b}, \eta^{G}_{\mathfrak{a},b})$ is a triad.*

Proof. Clear from 2.1 B) and 1.10. □

(2.3) Proposition *Let $n \in \mathbb{Z}$ and suppose that $(\iota^G_{\mathfrak{a},b}, \eta^G_{\mathfrak{a},b})$ is a triad. Then, there is an exact sequence*

$$0 \longrightarrow {}^G H^1_{\langle b\rangle} \circ \big({}^G H^{n-1}_{\mathfrak{a}}/({}^G\Gamma_{\langle b\rangle} \circ {}^G H^{n-1}_{\mathfrak{a}})\big) \longrightarrow {}^G H^n_{\mathfrak{a}+\langle b\rangle} \longrightarrow {}^G\Gamma_{\langle b\rangle} \circ {}^G H^n_{\mathfrak{a}} \longrightarrow 0$$

in $\mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$.

Proof. If $\mathfrak{b} \subseteq R$ is a G-graded ideal then by abuse of language we denote the local cohomology functors ${}^G H^n_{\mathfrak{b}}$ and the torsion functor ${}^G\Gamma_{\mathfrak{b}}$ with respect to $\mathfrak{b}$ just by $H^n_{\mathfrak{b}}$ and $\Gamma_{\mathfrak{b}}$.

Let $n \in \mathbb{Z}$. We set $U^n := \mathcal{R}^n(\Gamma_{\mathfrak{a}}(\bullet)_b)$, $\iota^n := \mathcal{R}^n\iota$ and $\eta^n := \mathcal{R}^n\eta$. Exactness of $\bullet_b$ implies $U^n = H^n_{\mathfrak{a}}(\bullet)_b$, and so $b_{U^n(M)}$ is an isomorphism for every G-graded R-module M. Therefore, it holds $\Gamma_{\langle b\rangle} \circ U^n = 0$, and hence $U^n(M)$ is b-quasidivisible for every G-graded R-module M. This implies $H^1_{\langle b\rangle} \circ U^n = H^1_{\langle b\rangle} \circ (U^n/(\Gamma_{\langle b\rangle} \circ U^n)) = 0$ by 1.5.

The triad sequence associated with (ι, η) yields an exact sequence

$$H^{n-1}_{\mathfrak{a}} \xrightarrow{\eta^{n-1}} U^{n-1} \longrightarrow H^n_{\mathfrak{a}+\langle b\rangle} \xrightarrow{\iota^n} H^n_{\mathfrak{a}} \xrightarrow{\eta^n} U^n,$$

hence an exact sequence

$$0 \longrightarrow U^{n-1}/\operatorname{Im}(\eta^{n-1}) \longrightarrow H^n_{\mathfrak{a}+\langle b\rangle} \longrightarrow \operatorname{Ker}(\eta^n) \longrightarrow 0.$$

Applying local cohomology to the exact sequence

$$0 \longrightarrow \operatorname{Im}(\eta^{n-1}) \longrightarrow U^{n-1} \longrightarrow U^{n-1}/\operatorname{Im}(\eta^{n-1}) \longrightarrow 0$$

yields an exact sequence

$$0 = \Gamma_{\langle b\rangle} \circ U^{n-1} \to \Gamma_{\langle b\rangle} \circ (U^{n-1}/\operatorname{Im}(\eta^n)) \to H^1_{\langle b\rangle} \circ \operatorname{Im}(\eta^{n-1}) \to H^1_{\langle b\rangle} \circ U^{n-1} = 0,$$

hence

$$U^{n-1}/\operatorname{Im}(\eta^n) = \Gamma_{\langle b\rangle} \circ (U^{n-1}/\operatorname{Im}(\eta^n)) \cong H^1_{\langle b\rangle} \circ \operatorname{Im}(\eta^{n-1}).$$

It is easy to see that $\Gamma_{\langle b\rangle} \circ H^n_{\mathfrak{a}}$ is a subfunctor of $\operatorname{Ker}(\eta^n)$. As $\Gamma_{\langle b\rangle} \circ H^n_{\mathfrak{a}+\langle b\rangle} = H^n_{\mathfrak{a}+\langle b\rangle}$ it follows $\operatorname{Ker}(\eta^n) = \operatorname{Im}(\iota^n) = \Gamma_{\langle b\rangle} \circ \operatorname{Im}(\iota^n) = \Gamma_{\langle b\rangle} \circ \operatorname{Ker}(\eta^n)$, and thus $\Gamma_{\langle b\rangle} \circ H^n_{\mathfrak{a}} = \operatorname{Ker}(\eta^n)$. This implies $\operatorname{Im}(\eta^{n-1}) \cong H^{n-1}_{\mathfrak{a}}/(\Gamma_{\langle b\rangle} \circ H^{n-1}_{\mathfrak{a}})$, and putting everything together we get an exact sequence as desired. □

(2.4) Corollary *Let $n \in \mathbb{Z}$ and suppose that R has the ITI-property with respect to $\mathfrak{a}$ and the ITR-property with respect to $\langle b\rangle$. Then, there is an exact sequence*

$$0 \longrightarrow {}^G H^1_{\langle b\rangle} \circ {}^G H^{n-1}_{\mathfrak{a}} \longrightarrow {}^G H^n_{\mathfrak{a}+\langle b\rangle} \longrightarrow {}^G\Gamma_{\langle b\rangle} \circ {}^G H^n_{\mathfrak{a}} \longrightarrow 0$$

in $\mathsf{Hom}(\mathsf{GrMod}^G(R), \mathsf{GrMod}^G(R))$.

Proof. Clear from 1.10, 2.2, 2.3 and [13, III.4.4.10]. □

(2.5) Proposition *Let $A \subseteq R^{\mathrm{hom}}$ be a subset, let M be a G-graded R-module such that ${}^G\Gamma_{\langle B\rangle}(M)$ is A-quasidivisible for every subset $B \subseteq A$, and suppose that R has the ITI-property with respect to $\langle B\rangle_R$ for every $B \subseteq A$. Then, it holds ${}^G H^n_{\langle B\rangle}(M) = 0$ for every $n \in \mathbb{N}$ and every subset $B \subseteq A$.*

Proof. We prove the claim by induction on $r := \operatorname{Card}(A)$. If $r = 0$ it holds obviously. So, suppose that $r > 0$ and that the claim holds for strictly smaller values of r. Let

$a \in A$ and $n \in \mathbb{N}$. We set $\mathfrak{a} := \langle A\rangle_R$ and $\mathfrak{b} := \langle A \setminus \{a\}\rangle_R$. Then, by [13, III.4.2.3] and 2.4 there is an exact sequence

$$ {}^G H^1_{\langle a\rangle_R}({}^G H^{n-1}_{\mathfrak{b}}(M)) \longrightarrow {}^G H^n_{\mathfrak{a}}(M) \longrightarrow {}^G \Gamma_{\langle a\rangle_R}({}^G H^n_{\mathfrak{b}}(M)) $$

in $\mathsf{GrMod}^G(R)$. As ${}^G H^n_{\mathfrak{b}}(M) = 0$ by hypothesis and hence ${}^G\Gamma_{\langle a\rangle_R}({}^G H^n_{\mathfrak{b}}(M))$, it suffices to show ${}^G H^1_{\langle a\rangle}({}^G H^{n-1}_{\mathfrak{b}}(M)) = 0$. If $n > 1$ it holds ${}^G H^{n-1}_{\mathfrak{b}}(M) = 0$ by hypothesis, hence ${}^G H^1_{\langle a\rangle}({}^G H^{n-1}_{\mathfrak{b}}(M)) = 0$ and thus the claim. If $n = 1$ then ${}^G H^0_{\mathfrak{b}}(M)$ is a-quasidivisible by hypothesis, and by [13, III.4.2.3; 4.4.10] and 1.5 it follows ${}^G H^1_{\langle a\rangle}({}^G H^{n-1}_{\mathfrak{b}}(M)) = 0$, thus the claim. □

(2.6) We denote by $\operatorname{ara}_R(\mathfrak{a}) := \inf\{\operatorname{Card}(E) \mid E \subseteq R^{\mathrm{hom}} \wedge \sqrt{\langle E\rangle} = \sqrt{\mathfrak{a}}\}$ the arithmetic rank of $\mathfrak{a}$. Clearly, it holds $\operatorname{ara}_R(\mathfrak{a}) \geq \operatorname{ara}_{R_{[\psi]}}(\mathfrak{a}_{[\psi]})$.

(2.7) Proposition *If R has the ITI-property with respect to every finitely generated ideal of R, then for every $i \in \mathbb{Z}$ with $i > \operatorname{ara}_R(\mathfrak{a})$ it holds ${}^G H^i_{\mathfrak{a}} = 0$.*

Proof. If $r := \operatorname{ara}_R(\mathfrak{a})$ is not finite then this is obvious. Otherwise, keeping in mind that ${}^G H^i_{\mathfrak{a}} = {}^G H^i_{\sqrt{\mathfrak{a}}}$ for every $i \in \mathbb{Z}$ we can suppose that there is a finite subset $E \subseteq R^{\mathrm{hom}}$ with $\langle E\rangle_R = \mathfrak{a}$ and $\operatorname{Card}(E) = r$, and show the claim by induction on r. If $r = 0$ it holds obviously. So, let $r > 0$ and suppose that ${}^G H^j_{\mathfrak{b}} = 0$ for every G-graded ideal $\mathfrak{b} \subseteq R$ and every $j \in \mathbb{Z}$ with $j > \operatorname{ara}_R(\mathfrak{b})$. Let $i \in \mathbb{Z}_{>r}$, let $e \in E$, and let $\mathfrak{b} := \langle E \setminus \{e\}\rangle_R$. Then, by 2.4 we have an exact sequence

$$ 0 \longrightarrow {}^G H^1_{\langle e\rangle_R} \circ {}^G H^{i-1}_{\mathfrak{b}} \longrightarrow {}^G H^i_{\mathfrak{a}} \longrightarrow {}^G\Gamma_{\langle e\rangle_R} \circ {}^G H^i_{\mathfrak{b}}. $$

As $\operatorname{ara}_R(\mathfrak{b}) < r < i$ we get ${}^G H^i_{\mathfrak{b}} = 0 = {}^G H^{i-1}_{\mathfrak{b}}$, hence ${}^G\Gamma_{\langle e\rangle_R} \circ {}^G H^i_{\mathfrak{b}} = 0$, and therefore ${}^G H^i_{\mathfrak{a}} \cong {}^G H^1_{\langle e\rangle_R} \circ {}^G H^{i-1}_{\mathfrak{b}} = 0$. Thus, the claim is proven. □

3. Torsion-faithful functors

Based on the notion of torsion-faithful functors and δ-functor techniques, we prove along the lines of [4, 1.9–1.13] a graded version of a general result on commutation of local cohomology functors with exact functors.

Throughout this section let $f : R \to R'$ be a morphism in GrAnn^G, let $\mathfrak{a} \subseteq R$ be a G-graded ideal, let $\mathfrak{a}' := \langle f(\mathfrak{a})\rangle_{R'}$, and let $E : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R')$ and $E' : \mathsf{GrMod}^G(R') \to \mathsf{GrMod}^G(R)$ be f-linear functors.

(3.1) The functor E is called $\mathfrak{a}$-*torsion-faithful* if ${}^G\Gamma_{\mathfrak{a}'} \circ E \circ {}^G\Gamma_{\mathfrak{a}} = E \circ {}^G\Gamma_{\mathfrak{a}}$, and the functor E' is called $\mathfrak{a}$-*torsion-faithful* if ${}^G\Gamma_{\mathfrak{a}} \circ E \circ {}^G\Gamma_{\mathfrak{a}'} = E \circ {}^G\Gamma_{\mathfrak{a}'}$.

(3.2) Lemma *a) Suppose that E is leftexact and that whenever M is a G-graded R-module with $M = {}^G\Gamma_{\mathfrak{a}}(M)$ and $x \in E(M)$, then there exists a finitely generated G-graded sub-R-module $N \subseteq M$ such that $x \in E(N)$. Then, E is $\mathfrak{a}$-torsion-faithful.*

b) Suppose that E' is leftexact and that whenever M is a G-graded R'-module with $M = {}^G\Gamma_{\mathfrak{a}'}(M)$ and $x \in E'(M)$, then there exists a finitely generated G-graded sub-R'-module $N \subseteq M$ such that $x \in E'(N)$. Then, E' is $\mathfrak{a}$-torsion-faithful.

Proof. a) Let M be a G-graded R-module, and let $x \in E({}^G\Gamma_{\mathfrak{a}}(M))$. By our hypothesis there exists a finitely generated G-graded sub-R-module $N \subseteq {}^G\Gamma_{\mathfrak{a}}(M)$ with $x \in E(N)$. Then, it holds ${}^G\Gamma_{\mathfrak{a}}(N) = N$, and hence there is an $n \in \mathbb{N}_0$ with $\mathfrak{a}^n N = 0$. Therefore, for every $y \in \mathfrak{a}^n$ the multiplication morphism y_N is a zero morphism, and hence $E(y_N) = f(y)_{E(N)}$ is a zero morphism, too. This implies

$f(\mathfrak{a})^n E(N) = 0$, hence $(\mathfrak{a}')^n E(N) = 0$ and in particular $(\mathfrak{a}')^n x = 0$. Thus, it holds $x \in {}^G\Gamma_{\mathfrak{a}'}(E({}^G\Gamma_{\mathfrak{a}}(M)))$, and so the claim is proven.

b) Let M be a G-graded R'-module, and let $x \in E'({}^G\Gamma_{\mathfrak{a}'}(M))$. By our hypothesis there exists a finitely generated G-graded sub-R'-module $N \subseteq {}^G\Gamma_{\mathfrak{a}'}(M)$ with $x \in E'(N)$. Then, it holds ${}^G\Gamma_{\mathfrak{a}'}(N) = N$, and hence there is an $n \in \mathbb{N}_0$ with $(\mathfrak{a}')^n N = 0$. Therefore, for every $y \in \mathfrak{a}^n$ the multiplication morphism $f(y)_N$ is a zero morphism, and hence $E'(f(y)_N) = y_{E(N)}$ is a zero morphism, too. This implies $\mathfrak{a}^n E'(N) = 0$ and in particular $\mathfrak{a}^n x = 0$. Thus, it holds $x \in {}^G\Gamma_{\mathfrak{a}}(E({}^G\Gamma_{\mathfrak{a}'}(M)))$, and so the claim is proven. □

(3.3) Corollary *a) If E is leftexact and commutes with filtering inductive limits, then E is $\mathfrak{a}$-torsion-faithful.*

b) If E' is leftexact and commutes with filtering inductive limits, then E' is $\mathfrak{a}$-torsion-faithful.

Proof. Clear from 3.2, since every G-graded R-module is a filtering inductive limit of finitely generated G-graded R-module. □

(3.4) Proposition *Let $A \subseteq R^{\text{hom}}$ be a finite subset with $\mathfrak{a} = \langle A \rangle_R$.*

a) Suppose that E is exact and $\langle a \rangle_R$-torsion-faithful for every $a \in A$. Then, it holds $E \circ {}^G\Gamma_{\mathfrak{a}} = {}^G\Gamma_{\mathfrak{a}'} \circ E$, and E is $\mathfrak{a}$-torsion-faithful.

b) Suppose that E' is exact and $\langle a \rangle_R$-torsion-faithful for every $a \in A$. Then, it holds $E' \circ {}^G\Gamma_{\mathfrak{a}'} = {}^G\Gamma_{\mathfrak{a}} \circ E'$, and E' is $\mathfrak{a}$-torsion-faithful.

Proof. a) First, let $a \in R^{\text{hom}}$, and suppose that $A = \{a\}$. Let M be a G-graded R-module, and consider the exact sequence

$$0 \to {}^G\Gamma_{\mathfrak{a}}(M) \to M \to M_a$$

in $\mathsf{GrMod}^G(R)$. Since E is leftexact, applying ${}^G\Gamma_{\mathfrak{a}'} \circ E \hookrightarrow E$ yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & E({}^G\Gamma_{\mathfrak{a}}(M)) & \hookrightarrow & E(M) & \longrightarrow & E(M_a) \\
 & & {\scriptstyle u}\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & {}^G\Gamma_{\mathfrak{a}'}(E({}^G\Gamma_{\mathfrak{a}}(M))) & \underset{v}{\hookrightarrow} & {}^G\Gamma_{\mathfrak{a}'}(E(M)) & \longrightarrow & {}^G\Gamma_{\mathfrak{a}'}(E(M_a))
\end{array}$$

in $\mathsf{GrMod}^G(R')$. Then, $\mathfrak{a}$-torsion-faithfulness of E implies that u is an isomorphism. Moreover, the multiplication morphism a_{M_a} is an isomorphism, hence $E(a_{M_a}) = f(a)_{E(M_a)}$ is an isomorphism, and so we have ${}^G\Gamma_{\mathfrak{a}'}(E(M_a)) = 0$. Therefore, v is an isomorphism, and thus it holds ${}^G\Gamma_{\mathfrak{a}'}(E(M))) = E({}^G\Gamma_{\mathfrak{a}}(M))$.

Next, we prove the claim. By the above we have ${}^G\Gamma_{\langle f(a) \rangle_{R'}} \circ E = E \circ {}^G\Gamma_{\langle a \rangle_R}$ for every $a \in A$. As E is rightexact it commutes with finite intersections, and so it follows

$$E \circ {}^G\Gamma_{\mathfrak{a}} = E \circ \Big(\bigcap_{a \in A} {}^G\Gamma_{\langle a \rangle_R}\Big) = \bigcap_{a \in A} (E \circ {}^G\Gamma_{\langle a \rangle_R}) =$$

$$\bigcap_{a \in A} ({}^G\Gamma_{\langle f(a) \rangle_{R'}} \circ E) = \Big(\bigcap_{a \in A} {}^G\Gamma_{\langle f(a) \rangle_{R'}}\Big) \circ E = {}^G\Gamma_{\mathfrak{a}'} \circ E,$$

hence in particular ${}^G\Gamma_{\mathfrak{a}'} \circ E \circ {}^G\Gamma_{\mathfrak{a}} = {}^G\Gamma_{\mathfrak{a}'} \circ {}^G\Gamma_{\mathfrak{a}'} \circ E = {}^G\Gamma_{\mathfrak{a}'} \circ E$. This proves the claim.

b) First, let $a \in R^{\rm hom}$, and suppose that $A = \{a\}$. Let M be a G-graded R'-module, and consider the exact sequence

$$0 \to {}^G\Gamma_{\mathfrak{a}'}(M) \to M \to M_{f(a)}$$

in $\mathsf{GrMod}^G(R')$. Since E' is leftexact, applying ${}^G\Gamma_{\mathfrak{a}} \circ E' \hookrightarrow E'$ yields a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & E'({}^G\Gamma_{\mathfrak{a}'}(M)) & \hookrightarrow & E'(M) & \longrightarrow & E'(M_{f(a)}) \\ & & u \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & {}^G\Gamma_{\mathfrak{a}}(E'({}^G\Gamma_{\mathfrak{a}'}(M))) & \overset{}{\underset{v}{\hookrightarrow}} & {}^G\Gamma_{\mathfrak{a}}(E'(M)) & \longrightarrow & {}^G\Gamma_{\mathfrak{a}}(E'(M_{f(a)})) \end{array}$$

in $\mathsf{GrMod}^G(R)$. Then, $\mathfrak{a}$-torsion-faithfulness of E' implies that u is an isomorphism. Moreover, the multiplication morphism $f(a)_{M_{f(a)}}$ is an isomorphism, hence $E'(f(a)_{M_{f(a)}}) = a_{E'(M_{f(a)})}$ is an isomorphism, and so we have ${}^G\Gamma_{\mathfrak{a}}(E'(M_{f(a)})) = 0$. Therefore, v is an isomorphism, and thus it holds ${}^G\Gamma_{\mathfrak{a}}(E'(M))) = E'({}^G\Gamma_{\mathfrak{a}'}(M))$.

Next, we prove the claim. By the above we have ${}^G\Gamma_{\langle a\rangle_R} \circ E' = E' \circ {}^G\Gamma_{\langle f(a)\rangle_{R'}}$ for every $a \in A$. As E' is rightexact it commutes with finite intersections, and so it follows

$$E \circ {}^G\Gamma_{\mathfrak{a}'} = E' \circ \big(\bigcap_{a\in A} {}^G\Gamma_{\langle f(a)\rangle_{R'}}\big) = \bigcap_{a\in A} (E' \circ {}^G\Gamma_{\langle f(a)\rangle_{R'}}) =$$

$$\bigcap_{a\in A} ({}^G\Gamma_{\langle a\rangle_R} \circ E') = \big(\bigcap_{a\in A} {}^G\Gamma_{\langle a\rangle_R}\big) \circ E' = {}^G\Gamma_{\mathfrak{a}} \circ E',$$

hence in particular ${}^G\Gamma_{\mathfrak{a}} \circ E' \circ {}^G\Gamma_{\mathfrak{a}'} = {}^G\Gamma_{\mathfrak{a}} \circ {}^G\Gamma_{\mathfrak{a}} \circ E' = {}^G\Gamma_{\mathfrak{a}} \circ E'$. This proves the claim. □

(3.5) Proposition *Let $a \in R^{\rm hom}$.*

a) Suppose that E is exact and $\langle a\rangle_R$-torsion-faithful, and let M be an a-quasidivisible G-graded R-module. Then, $E(M)$ is $f(a)$-quasidivisible.

b) Suppose that E' is exact and $\langle a\rangle_R$-torsion-faithful, and let M be an $f(a)$-quasidivisible G-graded R'-module. Then, $E'(M)$ is a-quasidivisible.

Proof. a) The multiplication morphism $a_{M/{}^G\Gamma_{\langle a\rangle_R}(M)}$ is an isomorphism, and so $E(a_{M/{}^G\Gamma_{\langle a\rangle_R}(M)}) = f(a)_{E(M/{}^G\Gamma_{\langle a\rangle_R}(M))}$ is an isomorphism, too. Since E is exact, 3.4 implies $E(M/{}^G\Gamma_{\langle a\rangle_R}(M)) \cong E(M)/{}^G\Gamma_{\langle f(a)\rangle_{R'}}(E(M))$, and herewith the claim is proven.

b) The multiplication morphism $f(a)_{M/{}^G\Gamma_{\langle f(a)\rangle_{R'}}(M)}$ is an isomorphism, and so $E'(f(a)_{M/{}^G\Gamma_{\langle f(a)\rangle_{R'}}(M)}) = a_{E'(M/{}^G\Gamma_{\langle f(a)\rangle_{R'}}(M))}$ is an isomorphism, too. Since E' is exact, 3.4 implies $E'(M/{}^G\Gamma_{\langle f(a)\rangle_{R'}}(M)) \cong E'(M)/{}^G\Gamma_{\langle a\rangle_R}(E'(M))$, and herewith the claim is proven. □

(3.6) Lemma *Let $A \subseteq R^{\rm hom}$ be a finite subset with $\mathfrak{a} = \langle A\rangle_R$, and let $n \in \mathbb{N}$.*

a) Let I be an injective G-graded R-module, and suppose that R has the ITI-property with respect to $\langle B\rangle_R$ for every $B \subseteq A$, and that E is exact and $\langle a\rangle_R$-torsion-faithful for every $a \in A$. Then, it holds ${}^GH^n_{\mathfrak{a}'}(E(I)) = 0$.

b) Let I be an injective G-graded R'-module, and suppose that R' has the ITI-property with respect to $\langle f(B)\rangle_{R'}$ for every $B \subseteq A$, and that E' is exact and $\langle a\rangle_R$-torsion-faithful for every $a \in A$. Then, it holds ${}^GH^n_{\mathfrak{a}}(E'(I)) = 0$.

Proof. a) Let $B \subseteq A$. Then, ${}^G\Gamma_{\langle B\rangle_R}(I)$ is injective and hence A-quasidivisible by 1.10. So, ${}^G\Gamma_{\langle f(B)\rangle_{R'}}(E(I)) = E({}^G\Gamma_{\langle B\rangle_R}(I))$ is $f(A)$-quasidivisible by 3.4 and 3.5, and then 2.5 implies the claim.

b) Let $B \subseteq A$. Then, ${}^G\Gamma_{\langle f(B)\rangle_{R'}}(I)$ is injective and hence $f(A)$-quasidivisible by 1.10. So, ${}^G\Gamma_{\langle B\rangle_R}(E'(I)) = E'({}^G\Gamma_{\langle f(B)\rangle_{R'}}(I))$ is A-quasidivisible by 3.4 and 3.5, and then 2.5 implies the claim. □

(3.7) Theorem *Let $A \subseteq R^{\mathrm{hom}}$ be a finite subset with $\mathfrak{a} = \langle A\rangle_R$.*

a) Suppose that R has the ITI-property with respect to $\langle B\rangle_R$ for every $B \subseteq A$, and that E is exact and $\langle a\rangle_R$-torsion-faithful for every $a \in A$. Then, there is an isomorphism of δ-functors

$$\big(E \circ {}^G H^n_{\mathfrak{a}}\big)_{n\in\mathbb{Z}} \xrightarrow{\cong} \big({}^G H^n_{\mathfrak{a}'} \circ E\big)_{n\in\mathbb{Z}}.$$

b) Suppose that R' has the ITI-proerty with respect to $\langle f(B)\rangle_{R'}$ for every $B \subseteq A$, and that E' is exact and $\langle a\rangle_R$-torsion-faithful for every $a \in A$. Then, there is an isomorphism of δ-functors

$$\big(E' \circ {}^G H^n_{\mathfrak{a}'}\big)_{n\in\mathbb{Z}} \xrightarrow{\cong} \big({}^G H^n_{\mathfrak{a}} \circ E'\big)_{n\in\mathbb{Z}}.$$

Proof. a) Let $B \subseteq A$, let $\mathfrak{b} := \langle B\rangle_R$ and let $\mathfrak{b}' := \langle f(B)\rangle_{R'}$. It follows from 3.4 that E is $\mathfrak{b}$-torsion-faithful and that $E \circ {}^G\Gamma_{\mathfrak{b}} = {}^G\Gamma_{\mathfrak{b}'} \circ E$, and the identity morphism of this functor can be extended uniquely to an isomorphism of δ-functors

$$\big(\mathcal{R}^n(E \circ {}^G\Gamma_{\mathfrak{b}})\big)_{n\in\mathbb{Z}} \xrightarrow{\cong} \big(\mathcal{R}^n({}^G\Gamma_{\mathfrak{b}'} \circ E)\big)_{n\in\mathbb{Z}}.$$

Since E is exact, the δ-functor on the left-hand side is canonically isomorphic to $(E \circ {}^G H^n_{\mathfrak{b}})_{n\in\mathbb{Z}}$, and the identity morphism of ${}^G\Gamma_{\mathfrak{b}'} \circ E$ can be uniquely extended to a morphism of δ-functors

$$\big(\mathcal{R}^n({}^G\Gamma_{\mathfrak{b}'} \circ E)\big)_{n\in\mathbb{Z}} \longrightarrow \big({}^G H^n_{\mathfrak{b}'} \circ E\big)_{n\in\mathbb{Z}}.$$

It suffices to show that this last morphism is an isomorphism. By [9, 2.2.1] and [13, III.2.4.8] this is the case if and only if ${}^G H^n_{\mathfrak{b}'}(E(I)) = 0$ for all $n \in \mathbb{N}$ and every injective G-graded R-module I. But this holds by 3.6, and so the claim is proven.

b) Let $B \subseteq A$, let $\mathfrak{b} := \langle B\rangle_R$ and let $\mathfrak{b}' := \langle f(B)\rangle_{R'}$. It follows from 3.4 that E' is $\mathfrak{b}$-torsion-faithful and that $E' \circ {}^G\Gamma_{\mathfrak{b}'} = {}^G\Gamma_{\mathfrak{b}} \circ E'$, and the identity morphism of this functor can be extended uniquely to an isomorphism of δ-functors

$$\big(\mathcal{R}^n(E' \circ {}^G\Gamma_{\mathfrak{b}'})\big)_{n\in\mathbb{Z}} \xrightarrow{\cong} \big(\mathcal{R}^n({}^G\Gamma_{\mathfrak{b}} \circ E')\big)_{n\in\mathbb{Z}}.$$

Since E' is exact, the δ-functor on the left-hand side is canonically isomorphic to $(E' \circ {}^G H^n_{\mathfrak{b}'})_{n\in\mathbb{Z}}$, and the identity morphism of ${}^G\Gamma_{\mathfrak{b}} \circ E'$ can be uniquely extended to a morphism of δ-functors

$$\big(\mathcal{R}^n({}^G\Gamma_{\mathfrak{b}} \circ E')\big)_{n\in\mathbb{Z}} \longrightarrow \big({}^G H^n_{\mathfrak{b}} \circ E'\big)_{n\in\mathbb{Z}}.$$

It suffices to show that this last morphism is an isomorphism. By [9, 2.2.1] and [13, III.2.4.8] this is the case if and only if ${}^G H^n_{\mathfrak{b}}(E'(I)) = 0$ for all $n \in \mathbb{N}$ and every injective G-graded R'-module I. But this holds by 3.6, and so the claim is proven. □

4. Degree restrictions and local cohomology

Stimulated by [4, 1.9–1.13] we apply now the commutation result from the last section in order to derive a condition for local cohomology to commute with restriction of degrees.

(4.1) We can consider the G-graded subring $f : (R_{(\varphi)})^{(\varphi)} \hookrightarrow R$ and the functor

$$E'(\bullet) := (\bullet_{(\varphi)})^{(\varphi)} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G((R_{(\varphi)})^{(\varphi)})$$

(see [13, 1.4]), and then we are in the situation of Section 3. The functor E' is obviously f-linear and exact. Moreover, if $\mathfrak{a} \subseteq (R_{(\varphi)})^{(\varphi)}$ is a G-graded ideal then E' is $\mathfrak{a}$-torsion-faithful, as is easily checked. In particular, it follows from 3.5 that E' preserves a-quasidivisibility whenever $a \in R^{\text{hom}}$ with $\deg(a) \in \varphi(F)$.

(4.2) Theorem *Let $A \subseteq \bigcup_{g\in\varphi(F)} R_g$ be a finite subset, let $\mathfrak{a} := \langle A\rangle_{R_{(\varphi)}}$ and let $\mathfrak{a}' := \langle A\rangle_R$. Moreover, suppose that R has the ITI-property with respect to $\langle B\rangle_R$ for every $B \subseteq A$. Then, there is an isomorphism of δ-functors*

$$\big(({}^G H^n_{\mathfrak{a}'}(\bullet)_{(\varphi)})^{(\varphi)}\big)_{n\in\mathbb{Z}} \xrightarrow{\cong} \big({}^G H^n_{\mathfrak{a}^{(\varphi)}}((\bullet_{(\varphi)})^{(\varphi)})\big)_{n\in\mathbb{Z}}.$$

Proof. Clear from 3.7 and 4.1. □

(4.3) Corollary *Let $A \subseteq R^{\text{hom}}$ be a finite subset, let $\mathfrak{a} := \langle A\rangle_R$, and suppose that R has the ITI-property with respect to $\langle B\rangle_R$ for every $B \subseteq A$. Moreover, suppose that* $\operatorname{Coker}(\varphi)$ *is finite, let $n \in \mathbb{N}$ be such that $n\deg(a) \in \varphi(F)$ for every $a \in A$, and let $\overline{\mathfrak{a}} := \langle a^n \mid a \in A\rangle_{R_{(\varphi)}}$. Then, there is an isomorphism of δ-functors*

$$\big(({}^G H^n_{\mathfrak{a}}(\bullet)_{(\varphi)})^{(\varphi)}\big)_{n\in\mathbb{Z}} \xrightarrow{\cong} \big({}^G H^n_{\overline{\mathfrak{a}}^{(\varphi)}}((\bullet_{(\varphi)})^{(\varphi)})\big)_{n\in\mathbb{Z}}.$$

Proof. Since $\sqrt{\langle\overline{\mathfrak{a}}\rangle_R} = \mathfrak{a}$ this follows from 4.2. □

5. Associated primes and non-zero-divisors

In this section we collect some auxiliary results on associated prime ideals of graded modules. By a well-known trick we can avoid Noetherian hypotheses on the ring, but rather demand the modules in question to be Noetherian. Most of the results in this section are inspired by [15] and [3, IV.1 Exercise 17].

(5.1) A) Let $\mathfrak{p} \subseteq R$ be a G-graded ideal. Then, $\mathfrak{p}$ is called *prime* if $R^{\text{hom}} \setminus \mathfrak{p}$ is multiplicatively closed. If $\mathfrak{p}_{[\psi]}$ is prime then so is $\mathfrak{p}$. The converse holds if G is torsionfree, as follows from [5, 17.25], keeping in mind that G is torsionfree if and only if there exists a structure of totally ordered group on G.

B) Let M be a G-graded R-module. A prime G-graded ideal $\mathfrak{p} \subseteq R$ is called *associated with* M if there exists $x \in M^{\text{hom}}$ with $\mathfrak{p} = (0 :_R x)$, and we denote by $\operatorname{Ass}_R(M)$ the set of prime G-graded ideals of R that are associated with M. If G is torsionfree then there is a bijection

$$\operatorname{Ass}_R(M) \xrightarrow{\cong} \operatorname{Ass}_{R_{[\psi]}}(M_{[\psi]}),\ \mathfrak{p} \mapsto \mathfrak{p}_{[\psi]},$$

as follows from A) and [5, 17.29], keeping in mind that G is torsionfree if and only if there exists a structure of totally ordered group on G.

(5.2) Lemma *Let M be a G-graded R-module, let $\mathfrak{p} \in \operatorname{Ass}_R(M)$, let $t \in M^{\text{hom}}$ be such that $\mathfrak{p} = (0 : Rt)$, and let $0 \neq N \subseteq \langle t\rangle_R$ be a G-graded sub-R-module. Then, it holds* $\operatorname{Ass}_R(N) = \{\mathfrak{p}\}$.

Proof. As $N^{\text{hom}} \setminus 0$ is nonempty it suffices to show that for every $u \in N^{\text{hom}} \setminus 0$ it holds $(0 :_R u) = \mathfrak{p}$. So, let $u \in N^{\text{hom}} \setminus 0$. There is an $r \in R^{\text{hom}}$ with $rt = u$, implying $(0 :_R t) \subseteq (0 :_R u)$, and since $rt = u \neq 0$ we moreover have $r \notin \mathfrak{p}$. For $x \in (0 :_R u)^{\text{hom}}$ it holds $xrt = 0$, hence $xr \in \mathfrak{p}$ and therefore $x \in \mathfrak{p}$, implying $(0 :_R u) \subseteq (0 :_R t)$ and therefore $(0 :_R u) = \mathfrak{p}$ as desired. □

(5.3) Proposition *Let M be a G-graded R-module and let $N \subseteq M$ be a G-graded sub-R-module. Then, it holds*

$$\text{Ass}_R(N) \subseteq \text{Ass}_R(M) \subseteq \text{Ass}_R(N) \cup \text{Ass}_R(M/N).$$

Proof. The first inclusion holds obviously. Let $\mathfrak{p} \in \text{Ass}_R(M)$. There is an $x \in M^{\text{hom}}$ with $\mathfrak{p} = (0 :_R x)$. Setting $L := \langle x \rangle_R$ it follows from 5.2 that $\mathfrak{p}$ is associated with every G-graded sub-R-module of L different from 0. Therefore, if $L \cap N \neq 0$ then we have $\mathfrak{p} \in \text{Ass}_R(L \cap N) \subseteq \text{Ass}_R(N)$ by the above. If $L \cap N = 0$ then the canonical epimorphism $M \twoheadrightarrow M/N$ induces by restriction a monomorphism $L \rightarrowtail M/N$, and the above implies $\mathfrak{p} \in \text{Ass}_R(L) \subseteq \text{Ass}_R(M/N)$. □

(5.4) Proposition *Let M be a G-graded R-module, and let $\mathfrak{p}$ a maximal element of the set $\{(0 :_R x) \mid x \in M^{\text{hom}} \setminus 0\}$. Then, it holds $\mathfrak{p} \in \text{Ass}_R(M)$.*

Proof. There is a $z \in M^{\text{hom}} \setminus 0$ with $\mathfrak{p} = (0 :_R z)$, and it suffices to show that $R^{\text{hom}} \setminus \mathfrak{p}$ is multiplicatively closed. Let $x, y \in R^{\text{hom}} \setminus \mathfrak{p}$, and assume that $xy \in \mathfrak{p}$. Then, we have $xyz = 0$, hence $x \in (0 :_R yz)$. As $(0 :_R z) \subseteq (0 :_R yz)$ and $x \notin \mathfrak{p}$ it follows $\mathfrak{p} \subsetneqq (0 :_R yz)$, and maximality of $\mathfrak{p}$ implies $yz = 0$, hence $y \in \mathfrak{p}$ as desired. □

(5.5) Proposition *Let M be a Noetherian G-graded R-module. Then,* $\text{Ass}_R(M)$ *is finite.*

Proof. Let $\mathbb{M}$ denote the set of G-graded sub-R-modules $N \subseteq M$ with the property that $\text{Ass}_R(M/N)$ is infinite, and assume that $\text{Ass}_R(M)$ is infinite. Since $0 \in \mathbb{M}$, Noetherianity of M yields the existence of a maximal element Q of $\mathbb{M}$. Let $\overline{M} := M/Q$, let $p : M \twoheadrightarrow \overline{M}$ denote the canonical epimorphism in $\mathsf{GrMod}^G(R)$, and let $\mathfrak{p} \in \text{Ass}_R(M/Q)$. By 5.1 B) there is an $x \in M^{\text{hom}} \setminus Q$ with $\mathfrak{p} = (0 :_R p(x))$, and setting $L := \langle p(x) \rangle_R$ we get $\text{Ass}_R(L) = \{\mathfrak{p}\}$ by 5.2. Setting $P := Q + \langle x \rangle_R$ we get $M/P \cong \overline{M}/L$ and $Q \subsetneqq P$, and hence $\text{Ass}_R(\overline{M}/L) = \text{Ass}_R(M/P)$ is finite. Therefore, from 5.3 we get the contradiction that $\text{Ass}_R(\overline{M})$ is contained in the finite set $\text{Ass}_R(L) \cup \text{Ass}_R(\overline{M}/L) = \{\mathfrak{p}\} \cup \text{Ass}_R(\overline{M}/L)$ □

(5.6) Proposition *Let M be a G-graded R-module, and let $\mathfrak{a} \subseteq R$ be a G-graded ideal with $\mathfrak{a}M = 0$. Then, there is a bijection*

$$\text{Ass}_R(M) \xrightarrow{\cong} \text{Ass}_{R/\mathfrak{a}}(M),\ \mathfrak{p} \mapsto \mathfrak{p}/\mathfrak{a}.$$

Proof. For $\mathfrak{p} \in \text{Ass}_R(M)$ there is an $x \in M^{\text{hom}}$ with $\mathfrak{p} = (0 :_R x)$, and it holds $\mathfrak{a} \subseteq (0 :_R x) = \mathfrak{p}$, hence $(0 :_{R/\mathfrak{a}} x) = (0 :_R x)/\mathfrak{a} = \mathfrak{p}/\mathfrak{a}$ is a prime G-graded ideal of $R/\mathfrak{a}$ and therefore it holds $\mathfrak{p}/\mathfrak{a} \in \text{Ass}_{R/\mathfrak{a}}(M)$, showing that the map in question exists. For a prime G-graded ideal $\mathfrak{p} \subseteq R$ with $\mathfrak{a} \subseteq \mathfrak{p}$ and $\mathfrak{p}/\mathfrak{a} \in \text{Ass}_{R/\mathfrak{a}}(M)$ there is an $x \in M^{\text{hom}}$ with $\mathfrak{p}/\mathfrak{a} = (0 :_{R/\mathfrak{a}} x)$, hence $\mathfrak{p}/\mathfrak{a} = (0 :_R x)/\mathfrak{a}$, and it follows $\mathfrak{p} = (0 :_R x) + \mathfrak{a} = (0 :_R x)$, thus $\mathfrak{p} \in \text{Ass}_R(M)$, showing its surjectivity. As it is clearly injective, the claim is proven. □

(5.7) Proposition *The G-graded ring R is Noetherian if and only if there exists a faithful, Noetherian G-graded R-module.*

Proof. If R is Noetherian then R is a faithful, Noetherian G-graded R-module. Conversely, let M be a faithful, Noetherian G-graded R-module. Then, M has a finite homogeneous generating set E, and $M^{\oplus E}$ is Noetherian. Faithfulness implies that the morphism $R \to M^{\oplus E}$, $x \mapsto (xe)_{e\in E}$ in $\mathsf{GrMod}^G(R)$ is a monomorphism, and thus R is Noetherian. □

(5.8) Proposition *Let M be a G-graded R-module, and suppose that M or R is Noetherian. Then, it holds $\mathrm{ZD}_R(M) = (\bigcup \mathrm{Ass}_R(M)) \cap R^{\mathrm{hom}}$.*

Proof. Obviously it holds $(\bigcup \mathrm{Ass}_R(M)) \cap R^{\mathrm{hom}} \subseteq \mathrm{ZD}_R(M)$. First, we show the converse under the hypothesis that R is Noetherian. Let $x \in \mathrm{ZD}_R(M)$. There is a $y \in M^{\mathrm{hom}} \setminus 0$ with $xy = 0$, and $(0 :_R y)$ is a proper G-graded ideal of R containing x. As R is Noetherian the set $\{(0 :_R z) \mid z \in M^{\mathrm{hom}} \setminus 0 \wedge (0 :_R y) \subseteq (0 :_R z)\}$ has a maximal element, and by 5.4 this lies in $\mathrm{Ass}_R(M)$. So, we get $x \in (0 :_R y) \subseteq \bigcup \mathrm{Ass}_R(M)$ and thus the claim.

Next, suppose that M is Noetherian. As the ideal $(0 :_R M) \subseteq R$ is graded we can consider the G-graded ring $\overline{R} := R/(0 :_R M)$, and we can consider M canonically as a G-graded $\overline{R}$-module. Let $p : R \twoheadrightarrow \overline{R}$ denote the canonical epimorphism of G-graded rings. The G-graded $\overline{R}$-module M is faithful and Noetherian, and thus $\overline{R}$ is Noetherian by 5.7. Let $a \in \mathrm{ZD}_R(M)$. There is an $x \in M^{\mathrm{hom}} \setminus 0$ with $ax = 0$, and it holds $p(a)x = 0$, hence $p(a) \in \mathrm{ZD}_{\overline{R}}(M) = (\bigcup \mathrm{Ass}_{\overline{R}}(M)) \cap R^{\mathrm{hom}}$ by the above. By 5.6 there is a $\mathfrak{p} \in \mathrm{Ass}_R(M)$ with $p(a) \in \mathfrak{p}/(0 :_R M)$, implying $a \in a + (0 :_R M) \subseteq \mathfrak{p} + (0 :_R M) = \mathfrak{p}$ and thus $a \in \bigcup \mathrm{Ass}_R(M)$ as desired. □

(5.9) Proposition *Let R be a positively $\mathbb{Z}$-graded ring, let $\mathfrak{a} \subseteq R$ be a G-graded ideal with $\mathfrak{a} \subseteq R_+$, and let M be a Noetherian $\mathbb{Z}$-graded R-module. Then, it holds $\mathfrak{a} \cap \mathrm{NZD}_R(M) \neq \emptyset$ if and only if ${}^{\mathbb{Z}}\Gamma_{\mathfrak{a}}(M) = 0$.*

Proof. We assume that ${}^{\mathbb{Z}}\Gamma_{\mathfrak{a}}(M) = 0$ and $\mathfrak{a} \cap \mathrm{NZD}_R(M) = \emptyset$. This implies $\mathfrak{a}^{\mathrm{hom}} \subseteq \mathrm{ZD}_R(M)$, hence $\mathfrak{a}^{\mathrm{hom}} \subseteq \bigcup \mathrm{Ass}_R(M)$ by 5.8. As $\mathrm{Ass}_R(M)$ is finite by 5.5, homogeneous prime avoidance (see [6, 10.13]) implies the existence of a $\mathfrak{p} \in \mathrm{Ass}_R(M)$ with $\mathfrak{a} \subseteq \mathfrak{p}$. So, there is an $x \in M^{\mathrm{hom}} \setminus 0$ with $\mathfrak{a}x = 0$, yielding the contradiction $x \in {}^{\mathbb{Z}}\Gamma_{\mathfrak{a}}(M)$. Thus, together with 1.4 C) the claim is proven. □

6. Components of $\mathbb{Z}$-graded local cohomology

Following its proof in [7, 15.1.5] we give a slight generalisation of the fundamental result that the homogeneous components of $\mathbb{Z}$-graded local cohomology modules are finitely generated and vanish in large degrees.

(6.1) Theorem *Let R be a positively $\mathbb{Z}$-graded ring that has the ITR-property with respect to R_+, let M be a Noetherian $\mathbb{Z}$-graded R-module, and let $i \in \mathbb{Z}$. Then, it holds $\mathrm{end}({}^{\mathbb{Z}}H^i_{R_+}(M)) < \infty$, and the R_0-module ${}^{\mathbb{Z}}H^i_{R_+}(M)_n$ is Noetherian for all $n \in \mathbb{Z}$.*

Proof. For $i \notin \mathbb{N}_0$ this is clear. We suppose $i \in \mathbb{N}_0$ and prove the claim by induction on i. We set $\Gamma := {}^{\mathbb{Z}}\Gamma_{R_+}$ and $H^k := {}^{\mathbb{Z}}H^k_{R_+}$ for $k \in \mathbb{Z}$. As M is Noetherian the same holds for its graded sub-R-module $H^0(M)$, and hence the R_0-module $H^0(M)_n$ is Noetherian for every $n \in \mathbb{Z}$ by [13, III.3.3.6]. In particular, the R_+-torsion module

$H^0(M)$ is finitely generated, and this implies $\mathrm{end}(H^0(M)) < \infty$. So, the claim holds in case $i = 0$.

Let $i > 0$, and suppose the claim to hold for strictly smaller values of i. As R has the ITR-property with respect to R_+ it holds $H^i(M) \cong H^i(M/\Gamma(M))$ by [13, III.4.4.10], and as M is Noetherian it holds $\Gamma(M/\Gamma(M)) = 0$ by [6, 1.4 B)b)]. So, without loss of generality we can suppose $\Gamma(M) = 0$. Then, 5.9 implies that there are $t \in \mathbb{N}$ and $x \in R_t \cap \mathrm{NZD}_R(M)$, and applying local cohomology to the exact sequence

$$0 \longrightarrow M(-t) \xrightarrow{x\cdot} M \longrightarrow M/xM \longrightarrow 0$$

in $\mathsf{GrMod}^{\mathbb{Z}}(R)$ yields for every $n \in \mathbb{Z}$ an exact sequence

$$H^{i-1}(M/xM)_n \longrightarrow H^i(M)_{n-t} \xrightarrow{x\cdot} H^i(M)_n$$

in $\mathsf{Mod}(R_0)$. By hypothesis there exists $n_0 \in \mathbb{Z}$ such that for every $n \in \mathbb{Z}_{\geq n_0}$ it holds $H^{i-1}(M/xM)_n = 0$. Therefore, $H^i(M)_{n-t} \xrightarrow{x\cdot} H^i(M)_n$ is a monomorphism for every $n \in \mathbb{Z}_{\geq n_0}$. Since $H^i(M)$ is an R_+-torsion module it follows $H^i(M)_n = 0$ for every $n \in \mathbb{Z}_{\geq n_0 - t}$, thus $\mathrm{end}(H^i(M)) < \infty$.

Finally, let $n \in \mathbb{Z}$. It remains to show Noetherianity of the R_0-module $H^i(M)_n$. There is a $k \in \mathbb{N}_0$ with $n + kt \geq n_0 - t$, and for every $j \in [0, k-1]$ we have the exact sequence

$$H^{i-1}(M/xM)_{n+(j+1)t} \longrightarrow H^i(M)_{n+jt} \longrightarrow H^i(M)_{n+(j+1)t}$$

in $\mathsf{Mod}(R_0)$, where $H^{i-1}(M/xM)_{n+(j+1)t}$ is Noetherian by hypothesis. So, it suffices to show that there is a $j \in [0, k-1]$ such that the R_0-module $H^i(M)_{n+(j+1)t}$ is Noetherian. But as $n + kt \geq n_0 - t$ and hence $H^i(M)_{n+kt} = 0$ this is fulfilled, and so the claim is proven. □

(6.2) By means of 6.1 we are able to define cohomological Hilbert functions of Noetherian $\mathbb{Z}$-graded modules over positively $\mathbb{Z}$-graded rings with the ITR-property with respect to R_+ and Artinian base rings.

7. Coarsening of local cohomology

In this section we prove by general δ-functor techniques that graded local cohomology commutes with coarsening functors, if the kernel of the coarsening morphism is finite. This result, in particular 7.2 and 7.3, are inspired by [10, I.2.10]. It allows to reduce the study of graded local cohomology to the case of torsionfree groups of degrees.

(7.1) According to [13, III.2.3.4] there is a canonical monomorphism

$$h_\psi : {}^G\mathrm{Hom}_R(\bullet, \blacksquare)_{[\psi]} \rightarrowtail {}^H\mathrm{Hom}_{R_{[\psi]}}(\bullet_{[\psi]}, \blacksquare_{[\psi]})$$

of contra-covariant bifunctors from $\mathsf{GrMod}^G(R)^2$ to $\mathsf{GrMod}^H(R_{[\psi]})$, given by

$$h_\psi(M, N) : {}^G\mathrm{Hom}_R(M, N)_{[\psi]} \rightarrowtail {}^H\mathrm{Hom}_{R_{[\psi]}}(M_{[\psi]}, N_{[\psi]}),\ u \mapsto u_{[\psi]}$$

for G-graded R-modules M and N.

(7.2) Proposition *Let M and N be G-graded R-modules. Then, it holds*

$${}^G\mathrm{Hom}_R(M, N) = \left\{ f \in \mathrm{Hom}_R(M, N) \;\middle|\; \begin{array}{c} \exists S \subseteq G : S \text{ is finite} \wedge \\ \forall g \in G : f(M_g) \subseteq \sum_{h \in S} N_{g+h} \end{array} \right\}.$$

Proof. Let $f \in {}^G\mathrm{Hom}_R(M,N)$. Then, $S := \mathrm{degsupp}(f)$ is a finite subset of G, and it holds $f = \sum_{h\in S} f_h$ with $f_h \in \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M, N(h))$ for every $h \in S$. So, for $g \in G$ we have $f(M_g) = (\sum_{h\in S} f_h)(M_g) \subseteq \sum_{h\in S} f_h(M_g) \subseteq \sum_{h\in S} N_{g+h}$.

Conversely, let $f \in \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M,N)$, and suppose that there is a finite subset $S \subseteq G$ such that for every $g \in G$ it holds $f(M_g) \subseteq \sum_{h\in S} N_{g+h}$. Let $h \in S$. For $g \in G$ we have the morphism $f_{h,g} : M_g \to N_{g+h},\ x \mapsto f(x)_{g+h}$ in $\mathsf{Mod}(R_0)$, and the family $(f_{h,g})_{g\in G}$ defines a morphism $f_h : M \to N(h)$ in $\mathsf{GrMod}^G(R)$, that is, $f_h \in {}^h\mathrm{Hom}_R(M,N)$. For $g \in G$ and $x \in M_g$ it holds $(\sum_{h\in S} f_h)(x) = \sum_{h\in S} f_{h,g}(x) = \sum_{h\in S} f(x)_{g+h} = f(x)$. This implies $f = \sum_{h\in F} f_h$ and therefore $f \in {}^G\mathrm{Hom}_R(M,N)$. □

(7.3) Proposition *Suppose that* $\mathrm{Ker}(\psi)$ *is finite. Then, the canonical monomorphism*

$$h_\psi : {}^G\mathrm{Hom}_R(\bullet, \blacksquare)_{[\psi]} \rightarrowtail {}^H\mathrm{Hom}_{R_{[\psi]}}(\bullet_{[\psi]}, \blacksquare_{[\psi]})$$

is an isomorphism.

Proof. Let $f \in {}^H\mathrm{Hom}_{R_{[\psi]}}(M_{[\psi]}, N_{[\psi]})$. By 7.2 there is a finite subset $S \subseteq H$ such that for every $h \in H$ it holds $f((M_{[\psi]})_h) \subseteq \sum_{l\in S}(N_{[\psi]})_{h+l}$. As there is a bijection from $\psi^{-1}(l)$ onto $\mathrm{Ker}(\psi)$ for every $l \in S$ we see that $T := \psi^{-1}(S)$ is a finite subset of G. Let $g \in G$. Then, it holds

$$f(M_g) \subseteq f((M_{[\psi]})_{\psi(g)}) \subseteq \sum_{l\in S}(N_{[\psi]})_{\psi(g)+l} = \sum_{l\in T}(N_{[\psi]})_{\psi(g)+\psi(l)} =$$

$$\sum_{l\in T}\ \sum_{k\in\psi^{-1}(\psi(g+l))} N_k = \sum_{l\in T}\ \sum_{k\in g+l+\mathrm{Ker}(\psi)} N_k = \sum_{k\in T+\mathrm{Ker}(\psi)} N_{g+k} = \sum_{k\in T} N_{g+k},$$

and thus 7.2 implies $f \in {}^G\mathrm{Hom}_R(M,N)$ as desired. □

(7.4) According to [13, III.4.3.4], there is for every $i \in \mathbb{Z}$ a canonical morphism

$$h^i_\psi : {}^G\mathrm{Ext}^i_R(\bullet, \blacksquare)_{[\psi]} \longrightarrow {}^H\mathrm{Ext}^i_{R_{[\psi]}}(\bullet_{[\psi]}, \blacksquare_{[\psi]})$$

of contra-covariant bifunctors from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^H(R_{[\psi]})$ with $h^0_\psi = h_\psi$ (see 7.1). These induce for every G-graded R-module M a morphism

$$(h^i_\psi(M, \blacksquare)) : \big({}^G\mathrm{Ext}^i_R(M, \blacksquare)_{[\psi]}\big)_{i\in\mathbb{Z}} \longrightarrow \big({}^H\mathrm{Ext}^i_{R_{[\psi]}}(M_{[\psi]}, \blacksquare_{[\psi]})\big)_{i\in\mathbb{Z}}$$

of δ-functors from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^H(R_{[\psi]})$ and a morphism

$$(h^i_\psi(\bullet, M)) : \big({}^G\mathrm{Ext}^i_R(\bullet, M)_{[\psi]}\big)_{i\in\mathbb{Z}} \longrightarrow \big({}^H\mathrm{Ext}^i_{R_{[\psi]}}(\bullet_{[\psi]}, M_{[\psi]})\big)_{i\in\mathbb{Z}}$$

of contravariant δ-functors from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^H(R_{[\psi]})$. Moreover, h^0_ψ is an isomorphism if and only if h^i_ψ is an isomorphism for every $i \in \mathbb{Z}$ (see [13, III.4.3.5]).

(7.5) Corollary *Suppose that* $\mathrm{Ker}(\psi)$ *is finite, and let* $i \in \mathbb{Z}$. *Then, the canonical morphism*

$$h^i_\psi : {}^G\mathrm{Ext}^i_R(\bullet, \blacksquare)_{[\psi]} \longrightarrow {}^H\mathrm{Ext}^i_{R_{[\psi]}}(\bullet_{[\psi]}, \blacksquare_{[\psi]})$$

is an isomorphism.

Proof. Clear from 7.3 and 7.4. □

(7.6) Corollary *Suppose that* $\mathrm{Ker}(\psi)$ *is finite, and let* M *be a* G*-graded* R*-module. Then, the canonical morphisms of* δ*-functors*

$$(h^i_\psi(M, \blacksquare)) : \left({}^G\mathrm{Ext}^i_R(M, \blacksquare)_{[\psi]}\right)_{i\in\mathbb{Z}} \longrightarrow \left({}^H\mathrm{Ext}^i_{R_{[\psi]}}(M_{[\psi]}, \blacksquare_{[\psi]})\right)_{i\in\mathbb{Z}}$$

and

$$(h^i_\psi(\bullet, M)) : \left({}^G\mathrm{Ext}^i_R(\bullet, M)_{[\psi]}\right)_{i\in\mathbb{Z}} \longrightarrow \left({}^H\mathrm{Ext}^i_{R_{[\psi]}}(\bullet_{[\psi]}, M_{[\psi]})\right)_{i\in\mathbb{Z}}$$

are isomorphisms.

Proof. Clear from 7.5 and 7.4. □

(7.7) Let J be a right filtering ordered set, and let $F : J \to \mathsf{GrMod}^G(R)$ be a projective system in $\mathsf{GrMod}^G(R)$ over J. According to [13, III.4.3.8] this gives rise to a universal δ-functor $(\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet))_{i\in\mathbb{Z}}$ from $\mathsf{GrMod}^G(R)$ to itself. Composition with coarsening yields a universal δ-functors $(\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet)_{[\psi]})_{i\in\mathbb{Z}}$ and an exact δ-functor $(\varinjlim_J {}^H\mathrm{Ext}^i_{R_{[\psi]}}(F_{[\psi]}, \bullet_{[\psi]}))_{i\in\mathbb{Z}}$ from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^H(R_{[\psi]})$. Furthermore, the morphisms $h^i_\psi(F(j), \bullet)$ for $i \in \mathbb{Z}$ and $j \in J$ (see 7.4) induce a morphism

$$(h^i_{F,\psi})_{i\in\mathbb{Z}} : \left(\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet)_{[\psi]}\right)_{i\in\mathbb{Z}} \longrightarrow \left(\varinjlim_J {}^H\mathrm{Ext}^i_{R_{[\psi]}}(F_{[\psi]}, \bullet_{[\psi]})\right)_{i\in\mathbb{Z}}$$

of δ-functors from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^H(R_{[\psi]})$.

(7.8) Theorem *Let* J *be a right filtering ordered set, let* $F : J \to \mathsf{GrMod}^G(R)$ *be a projective system in* $\mathsf{GrMod}^G(R)$ *over* J*, and suppose that* $\mathrm{Ker}(\psi)$ *is finite. Then, the canonical morphism of* δ*-functors*

$$(h^i_{F,\psi})_{i\in\mathbb{Z}} : \left(\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet)_{[\psi]}\right)_{i\in\mathbb{Z}} \longrightarrow \left(\varinjlim_J {}^H\mathrm{Ext}^i_{R_{[\psi]}}(F_{[\psi]}, \bullet_{[\psi]})\right)_{i\in\mathbb{Z}}$$

is an isomorphism.

Proof. We know from 7.5 that $h^i_\psi(F(j), \bullet)$ is an isomorphism for every $j \in J$ and every $i \in \mathbb{Z}$, and hence $\varinjlim_J h^i_\psi(F, \bullet)$ is an isomorphism for every $i \in \mathbb{Z}$. But universality of $(\varinjlim_J {}^G\mathrm{Ext}^i_R(F, \bullet)_{[\psi]})_{i\in\mathbb{Z}}$ implies $h^i_{F,\psi} = \varinjlim_J h^i_\psi(F, \bullet)$ for every $i \in \mathbb{Z}$, and thus the claim is proven. □

(7.9) Let J be a right filtering ordered set, and let $\mathfrak{A}$ be a projective system of G-graded ideal of R over J. This defines a projective system $R/\mathfrak{A}$ in $\mathsf{GrMod}^G(R)$ over R (see [13, III.3.5.3]). Now, the local cohomology functors with respect to $\mathfrak{A}$ and the higher ideal transformation functors with respect to $\mathfrak{A}$ are defined as the universal δ-functors

$$\left({}^GH^i_{\mathfrak{A}}(\bullet)\right)_{i\in\mathbb{Z}} := \left(\varinjlim_J {}^G\mathrm{Ext}^i_R(R/\mathfrak{A}, \bullet)\right)_{i\in\mathbb{Z}}$$

and

$$\left({}^GD^i_{\mathfrak{A}}\bullet\right)_{i\in\mathbb{Z}} := \left(\varinjlim_J {}^G\mathrm{Ext}^i_R(\mathfrak{A}, \bullet)\right)_{i\in\mathbb{Z}}.$$

Therefore, by 7.7 there are canonical morphisms of δ-functors

$$(h^i_{R/\mathfrak{A},\psi})_{i\in\mathbb{Z}} : \left({}^GH^i_{\mathfrak{A}}(\bullet)_{[\psi]}\right)_{i\in\mathbb{Z}} \longrightarrow \left({}^HH^i_{\mathfrak{A}_{[\psi]}}(\bullet_{[\psi]})\right)_{i\in\mathbb{Z}}$$

and

$$(h^i_{\mathfrak{A},\psi})_{i\in\mathbb{Z}} : \left({}^GD^i_{\mathfrak{A}}(\bullet)_{[\psi]}\right)_{i\in\mathbb{Z}} \longrightarrow \left({}^HD^i_{\mathfrak{A}_{[\psi]}}(\bullet_{[\psi]})\right)_{i\in\mathbb{Z}}.$$

(7.10) Corollary *Let J be a right filtering ordered set, let $\mathfrak{A}$ be a projective system of G-graded ideal of R over J, and suppose that* $\mathrm{Ker}(\psi)$ *is finite. Then, the canonical morphisms of δ-functors*

$$(h^i_{R/\mathfrak{A},\psi})_{i\in\mathbb{Z}} : \big({}^G H^i_{\mathfrak{A}}(\bullet)_{[\psi]}\big)_{i\in\mathbb{Z}} \longrightarrow \big({}^H H^i_{\mathfrak{A}_{[\psi]}}(\bullet_{[\psi]})\big)_{i\in\mathbb{Z}}$$

and

$$(h^i_{\mathfrak{A},\psi})_{i\in\mathbb{Z}} : \big({}^G D^i_{\mathfrak{A}}(\bullet)_{[\psi]}\big)_{i\in\mathbb{Z}} \longrightarrow \big({}^H D^i_{\mathfrak{A}_{[\psi]}}(\bullet_{[\psi]})\big)_{i\in\mathbb{Z}}$$

are isomorphisms.

Proof. Clear from 7.8 and 7.9. □

(7.11) Suppose that G is finitely generated. Then, there are $r \in \mathbb{N}_0$ and a finite subgroup $T \subseteq G$ such that $G = \mathbb{Z}^r \oplus T$. Let $\pi : G \twoheadrightarrow \mathbb{Z}^r$ denote the canonical projection, so that $\mathrm{Ker}(\pi) = T$ is finite. Now, let J be a right filtering ordered set, and let $\mathfrak{A}$ be a projective system of G-graded ideals of R over J. If M is a G-graded R-module, then 7.10 yields for every $i \in \mathbb{Z}$ canonical isomorphisms

$${}^G H^i_{\mathfrak{A}}(M)_{[\pi]} \cong {}^{\mathbb{Z}^r} H^i_{\mathfrak{A}_{[\pi]}}(M_{[\pi]})$$

and

$${}^G D^i_{\mathfrak{A}}(M)_{[\pi]} \cong {}^{\mathbb{Z}^r} D^i_{\mathfrak{A}_{[\pi]}}(M_{[\pi]})$$

of $\mathbb{Z}^r$-graded $R_{[\pi]}$-modules and thus allows in some sense to reduce the study of local cohomology with graduations over finitely generated groups to the study of local cohomology with graduations over free groups of finite rank.

8. Filterregular elements and sequences

Inspired by [4, 5.1], we collect some basic results on and characterisations of filter-regular elements, filter-regular sequences and saturated filter-regular sequences.

Throughout this section let $\mathfrak{a} \subseteq R$ be a G-graded ideal.

(8.1) A) An element $f \in R^{\mathrm{hom}}$ is called *$\mathfrak{a}$-filter-regular with respect to M* if $f \in \mathrm{NZD}_R(M/{}^G\Gamma_{\mathfrak{a}}(M))$, and a finite sequence $(f_i)_{i=1}^r$ in R^{hom} is called *$\mathfrak{a}$-filter-regular with respect to M* if f_i is $\mathfrak{a}$-filter-regular with respect to $M/\sum_{j=1}^{i-1} f_i M$ for every $i \in [1, r]$. Clearly, a finite sequence in R^{hom} is $\mathfrak{a}$-filter-regular with respect to M if and only if it is $\mathfrak{a}_{[\psi]}$-filter-regular with respect to $M_{[\psi]}$.

B) Clearly, every M-regular sequence in R^{hom} is $\mathfrak{a}$-filter-regular with respect to M. If ${}^G\Gamma_{\mathfrak{a}}(M) = 0$ then $f \in R^{\mathrm{hom}}$ is $\mathfrak{a}$-filter-regular with respect to M if and only if $f \in \mathrm{NZD}_R(M)$. Conversely, if ${}^G\Gamma_{\mathfrak{a}}(M) = M$ then every finite sequence in R^{hom} is $\mathfrak{a}$-filter-regular with respect to M.

C) Let $N \subseteq M$ be a G-graded sub-R-module. If $f \in R^{\mathrm{hom}}$ is $\mathfrak{a}$-filter-regular with respect to M then it is so with respect to N. If ${}^G\Gamma_{\mathfrak{a}}(N) = N$ then a finite sequence in R^{hom} is $\mathfrak{a}$-filter-regular with respect to M if and only if it is so with respect to M/N.

D) If $(f_i)_{i=1}^r$ is a finite sequence in R^{hom} then it is $\mathfrak{a}$-filter-regular with respect to M if and only if $(f_i^{n_i})_{i=1}^r$ is $\mathfrak{a}$-filter-regular with respect to M for every sequence $(n_i)_{i=1}^r$ in $\mathbb{N}$, and this holds if and only if there is a sequence $(n_i)_{i=1}^r$ in $\mathbb{N}$ such that $(f_i^{n_i})_{i=1}^r$ is $\mathfrak{a}$-filter-regular with respect to M.

(8.2) Lemma *Let $f \in R^{\mathrm{hom}}$. We consider the following statements:*

(1) f is $\mathfrak{a}$-filter-regular with respect to M;

(2) $({}^G\Gamma_{\mathfrak{a}}(M) :_M f) \subseteq {}^G\Gamma_{\mathfrak{a}}(M)$;

(3) $(0 :_M f) \subseteq {}^G\Gamma_{\mathfrak{a}}(M)$.

a) It holds (1)⇔(2)⇒(3).

b) If $\mathfrak{a}^n$ is finitely generated for some $n \in \mathbb{N}$ then it holds (1)⇔(2)⇔(3).

Proof. As $(0 :_M f) \subseteq ({}^G\Gamma_{\mathfrak{a}}(M) :_M f)$ claim a) holds obviously. So, suppose that $n \in \mathbb{N}$ is such that $\mathfrak{a}^n$ is finitely generated and that $(0 :_M f) \subseteq {}^G\Gamma_{\mathfrak{a}}(M)$, and let $x \in M$ be such that $fx \in {}^G\Gamma_{\mathfrak{a}}(M)$. There is an $m \in \mathbb{N}$ with $\mathfrak{a}^m fx = 0$, and hence it holds $\mathfrak{a}^m x \subseteq (0 :_M f) \subseteq {}^G\Gamma_{\mathfrak{a}}(M)$. Our hypothesis implies that there is an $l \in \mathbb{N}$ such that $\mathfrak{a}^l x = 0$, and therefore we get $x \in {}^G\Gamma_{\mathfrak{a}}(M)$, thus claim b). □

(8.3) A) Let A be a ring, and let T be an A-module. A prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ is called *weakly associated with* T if there is an $x \in T$ with $\mathfrak{p} \in \min(0 :_A x)$. We denote by $\mathrm{Ass}^{\mathrm{f}}_A(T)$ the set of prime ideals of A that are weakly associated with T. Clearly, it holds $\mathrm{Ass}_A(T) \subseteq \mathrm{Ass}^{\mathrm{f}}_A(T)$. Moreover, by [3, IV.1 Exercise 17] it holds $\mathrm{ZD}_A(T) = \bigcup \mathrm{Ass}^{\mathrm{f}}_A(T)$, if $S \subseteq A$ is a multiplicative subset, then it holds

$$\mathrm{Ass}^{\mathrm{f}}_{S^{-1}A}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}^{\mathrm{f}}_R(M) \wedge \mathfrak{p} \cap S = \emptyset\},$$

and if $0 \to T' \to T \to T'' \to 0$ is an exact sequence in $\mathsf{Mod}(A)$, then it holds

$$\mathrm{Ass}^{\mathrm{f}}_A(T') \subseteq \mathrm{Ass}^{\mathrm{f}}_A(T) \subseteq \mathrm{Ass}^{\mathrm{f}}_A(T'') \cup \mathrm{Ass}^{\mathrm{f}}_A(T').$$

Furthermore, if A is Noetherian then it holds $\mathrm{Ass}^{\mathrm{f}}_A(T) = \mathrm{Ass}_A(T)$. On use of 5.6 and 5.7 we see that if T is Noetherian then it also holds $\mathrm{Ass}^{\mathrm{f}}_A(T) = \mathrm{Ass}_A(T)$.[1]

B) Let G be torsionfree, and keep in mind that this implies that $\mathrm{Ass}_R(M) \subseteq {}^*\mathrm{Spec}(R) \subseteq \mathrm{Spec}(R)$. Moreover, from [3, IV.3 Exercise 1] we know that $\mathrm{Ass}^{\mathrm{f}}_R(M) \subseteq {}^*\mathrm{Spec}(R)$, and for every $\mathfrak{p} \in \mathrm{Ass}^{\mathrm{f}}_R(M)$ there is an $x \in M^{\mathrm{hom}}$ with $\mathfrak{p} \in \min(0 :_R x)$.

(8.4) Proposition *Let G be torsionfree and let $f \in R^{\mathrm{hom}}$. We consider the following statements:*

(1) f is $\mathfrak{a}$-filter-regular with respect to M;

(2) For every $\mathfrak{p} \in {}^*\mathrm{Spec}(R) \setminus \mathrm{Var}(\mathfrak{a})$ it holds $\frac{f}{1} \in \mathrm{NZD}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$;

(3) $f \notin \bigcup(\mathrm{Ass}^{\mathrm{f}}_R(M) \setminus \mathrm{Var}(\mathfrak{a}))$;

(4) $f \notin \bigcup(\mathrm{Ass}_R(M) \setminus \mathrm{Var}(\mathfrak{a}))$.

a) It holds (1)⇒(2)⇔(3)⇒(4).

b) If R is Noetherian or M is Noetherian then it holds (3)⇔(4).

c) If R and M are Noetherian then it holds (1)⇔(2)⇔(3)⇔(4).

Proof. a) Suppose that (1) holds, and let $\mathfrak{p} \in {}^*\mathrm{Spec}(R) \setminus \mathrm{Var}(\mathfrak{a})$. As $\mathrm{Supp}({}^G\Gamma_{\mathfrak{a}}(M))$ is contained in $\mathrm{Var}(\mathfrak{a})$ it holds $(M/{}^G\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}} = M_{\mathfrak{p}}$, and then (2) follows from exactness of $\bullet_{\mathfrak{p}}$. From 8.3 we know that (2) holds if and only if $\frac{f}{1} \notin \bigcup \mathrm{Ass}^{\mathrm{f}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for every $\mathfrak{p} \in {}^*\mathrm{Spec}(R) \setminus \mathrm{Var}(\mathfrak{a})$, and moreover that this holds if and only if

$$\tfrac{f}{1} \notin \bigcup\{\mathfrak{q}_{\mathfrak{p}} \mid \mathfrak{q} \in \mathrm{Ass}^{\mathrm{f}}_R(M) \wedge \mathfrak{q} \subseteq \mathfrak{p}\}$$

[1] This gives a new proof of 5.8.

for every $\mathfrak{p} \in {}^*\mathrm{Spec}(R) \setminus \mathrm{Var}(\mathfrak{a})$. But this is equivalent to f not being contained in

$$\bigcup_{\mathfrak{p} \in {}^*\mathrm{Spec}(R) \setminus \mathrm{Var}(\mathfrak{a})} \{\mathfrak{q} \in \mathrm{Ass}^{\mathrm{f}}_R(M) \mid \mathfrak{q} \subseteq \mathfrak{p}\},$$

and as this latter set equals $\bigcup \mathrm{Ass}^{\mathrm{f}}_R(M) \setminus \mathrm{Var}(\mathfrak{a})$ we see that (2) is equivalent to (3). Finally, (3) implies (4) by 8.3. b) is clear by 8.3. To show c), suppose that R and M are Noetherian. As $\mathrm{Ass}_R(M/{}^G\Gamma_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(M) \setminus \mathrm{Var}(\mathfrak{a})$ by [6, 1.9] and moreover

$$\mathrm{ZD}_R(M/{}^G\Gamma_{\mathfrak{a}}(M)) = \bigcup \mathrm{Ass}_R(M/{}^G\Gamma_{\mathfrak{a}}(M)) \cap R^{\mathrm{hom}},$$

we see that (4) implies (1). Thus, the claim holds by a) and b). □

(8.5) Corollary *Let G be torsionfree and let $(f_i)_{i=1}^r$ be a finite sequence in R^{hom}. We consider the following statements:*

(1) $(f_i)_{i=1}^r$ is $\mathfrak{a}$-filter-regular with respect to M;

(2) If $\mathfrak{p} \in {}^*\mathrm{Spec}(R) \setminus \mathrm{Var}(\mathfrak{a})$ then the sequence $(\frac{f_i}{1})_{i=1}^r$ in $R_{\mathfrak{p}}$ is $M_{\mathfrak{p}}$-regular.

a) It holds (1)⇒(2).

b) If R and M are Noetherian then it holds (1)⇔(2).

Proof. This follows from 8.4 a), c) on use of exactness of $\bullet_{\mathfrak{p}}$. □

(8.6) A finite sequence $(f_i)_{i=1}^r$ in R^{hom} that is $\mathfrak{a}$-filter-regular with respect to M is called *saturated with respect to* $\mathfrak{a}$ *and* M if

$${}^G\Gamma_{\mathfrak{a}}\big(M/\sum_{i=1}^r f_iM\big) = M/\sum_{i=1}^r f_iM.$$

(8.7) Lemma *If there is a finitely generated G-graded ideal $\mathfrak{b} \subseteq R$ with $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \sqrt{(0 :_R M)}$ then it holds ${}^G\Gamma_{\mathfrak{a}}(M) = M$.*

Proof. As $\mathfrak{b}$ is finitely generated there is an $n \in \mathbb{N}$ with $\mathfrak{b}^n \subseteq (0 :_R M)$, implying $\mathfrak{b}^n M = 0$ and therefore the claim. □

(8.8) Proposition *Let $\mathbf{f} = (f_i)_{i=1}^r$ be a finite sequence in R^{hom} that is $\mathfrak{a}$-filter-regular with respect to M. We consider the following statements:*

(1) $\mathbf{f}$ is saturated with respect to $\mathfrak{a}$ and M;

(2) It holds $\mathfrak{a} \subseteq \sqrt{(0 :_R M) + \mathbf{f}R}$;

(3) It holds $\mathfrak{a} \subseteq \sqrt{(0 :_R M/\mathbf{f}M)}$.

a) It holds (2)⇒(3).

b) If $\sqrt{(0 :_R M/\mathbf{f}M)}$ is finitely generated then it holds (2)⇒(3)⇒(1), and if $\sqrt{(0 :_R M) + \mathbf{f}R}$ is finitely generated, then it holds (2)⇒(1).

c) If $M/\mathbf{f}M$ is finitely generated then it holds (1)⇒(3).

d) If M is finitely generated then it holds (1)⇒(3)⇔(2).

e) If M is finitely generated and if moreover $\sqrt{(0 :_R M/\mathbf{f}M)}$ or $\sqrt{(0 :_R M) + \mathbf{f}R}$ is finitely generated[2], then it holds (1)⇔(3)⇔(2).

Proof. a) hold as $(0 :_R M) + \mathbf{f}R \subseteq (0 :_R M/\mathbf{f}M)$, and with the same argument 8.7 implies b). c) If $M/\mathbf{f}M$ is finitely generated and (1) holds then there is an $n \in \mathbb{N}$ with $\mathfrak{a}^n \subseteq (0 :_R M/\mathbf{f})$, and so it holds (3). Finally, d) is clear from a), c) and [5, 5.5], and e) is clear from b) and d). □

[2] in particular if R and M are Noetherian

9. Graded fields and graded local rings

In preparation for the next section we study graded analogues of fields and local rings. It turns out that every module over a graded field is free, and hence the machinery of linear algebra applies in these categories. Furthermore, we prove a graded version of Nakayama's Lemma. The results in this section are inspired by [4, Chapter 7], [2, VIII.6] and [11].

(9.1) A) If $x \in R^{\rm hom}$ is invertible then $x^{-1} \in R^{\rm hom}$ and $\deg(x^{-1}) = -\deg(x)$. The G-graded ring R is called a *field* if every element of $R^{\rm hom} \setminus 0$ is invertible. If this is the case then $D := \{g \in G \mid R_g \neq 0\} \subseteq G$ is a subgroup.

B) If R is a field then the F-graded ring $R_{(\varphi)}$ is a field. In particular, if R is a field then the ring R_0 is a field.

C) If R is a field then the H-graded ring $R_{[\psi]}$ is not necessarily a field. Indeed, if K is a field and R denotes the Laurent algebra over K in one indeterminate, furnished with its canonical $\mathbb{Z}$-graduation, then R is a field, but its underlying ring (that is, $R_{[\psi]}$ with $\psi : \mathbb{Z} \to 0$) is not. On the other hand, as $R^{\rm hom} \subseteq (R_{[\psi]})^{\rm hom}$ it is clear that if the H-graded ring $R_{[\psi]}$ is a field, then so is R. However, such a situation may impose some conditions on ψ (see 9.2, 9.3).

(9.2) Proposition *If the H-graded ring $R_{[\psi]}$ is a field and* $\mathrm{Ker}(\psi)$ *is torsionfree, then $(R_{[\psi]})_0 = R_0$.*

Proof. Let $G' := \{g \in G \mid R_g \neq 0\}$ and $H' := \{h \in H \mid (R_{[\psi]})_h \neq 0\}$, and let $i : G' \hookrightarrow G$ and $j : H' \hookrightarrow H$ denote the canonical injections. Then, ψ induces by restriction and coastriction an epimorphism $\psi' : G' \to H'$. Moreover, the G'-graded ring $R_{(i)}$ is a field, and the H'-graded ring $(R_{[\psi]})_{(j)} = (R_{(i)})_{[\psi']}$ is a field. Furthermore, $\mathrm{Ker}(\psi') \subseteq \mathrm{Ker}(\psi)$ is torsionfree, and so we can suppose without loss of generality that $G' = G$ and $H' = H$. Now, let $\vartheta : \mathrm{Ker}(\psi) \to 0$ denote the canonical morphism, induced by restriction and coastriction by ψ, and let $\iota : \mathrm{Ker}(\psi) \hookrightarrow G$ denote the canonical injection. The $\mathrm{Ker}(\psi)$-graded ring $R_{(\iota)}$ is a field, and the (ungraded) ring $(R_{(\iota)})_{[\vartheta]} = (R_{[\psi]})_0$ is a field. As $\mathrm{Ker}(\psi)$ is torsionfree, we may choose a structure of totally ordered group on $\mathrm{Ker}(\psi)$. Now, we assume the claim to be false. Then, there is a $g \in \mathrm{Ker}(\psi) \setminus 0$ and an $x \in R_g \setminus 0$. As $x^{-1} \in R_{-g}$ we can suppose without loss of generality that $g > 0$. As the (ungraded) ring $(R_{[\psi]})_0$ is a field, there is a $y \in (R_{[\psi]})_0$ such that $(1-a)b = 1$. There is a strictly increasing finite family $(g_i)_{i=1}^r$ in G such that $b = \sum_{i=1}^r b_{g_i}$ with $b_{g_i} \in R_{g_i} \setminus 0$ for $i \in [1, r]$, and it follows $1 = \sum_{i=1}^r b_{g_i} - ab_{g_i}$. But 1 being homogeneous and $g > 0$ yield the contradiction $g_1 = g + g_r > g_1$. Thus, the claim is proven. □

(9.3) Corollary *If G is torsionfree and the ring underlying R is a field, then it equals R_0.*

Proof. Clear from 9.2 with $\psi : G \twoheadrightarrow 0$. □

(9.4) In [11, 1.3.8–9] there are described (ungraded) fields that allow a G-graduation for the choices $G = \mathbb{Z}/n\mathbb{Z}$ with arbitrary $n \in \mathbb{N}$, and $G = \bigoplus_{i \in I} \mathbb{Z}/n_i\mathbb{Z}$ with an arbitrary countable family $(n_i)_{i \in I}$ in $\mathbb{N}$.

(9.5) Proposition *Suppose that R is a field, let M be a G-graded R-module, and let $L \subseteq S \subseteq M$ such that S is a homogeneous generating set of M and that L is free. Then, there exists a homogeneous basis B of M with $L \subseteq B \subseteq S$.*

Proof. The set of homogeneous free subsets of M, contained in S and containing L, is nonempty and inductive, hence has a maximal element B by Zorn's Lemma. It suffices to show $\langle B\rangle_R = M$. If this is not the case then there is a $x \in M^{\text{hom}} \setminus \langle B\rangle$, and in order to derive a contradiction it suffices to show that $B \cup \{x\}$ is free. So, let $\mu \in R$ and let $(\lambda_b)_{b\in B}$ be a family of finite support in R such that $\mu x + \sum_{b\in B} \lambda_b b = 0$. We have to show that $\mu = 0$ and $\lambda_b = 0$ for every $b \in B$. If $\mu = 0$ this is clear, for B is free. Otherwise, there is a $g \in G$ with $\mu_g \neq 0$, and it follows $\mu_g x + \sum_{b\in B}(\lambda_b)_{g+\deg(x)-\deg(b)} b = 0$, so we can suppose without loss of generality that μ is homogeneous. Then, it is invertible, for R is a field, and this implies the contradiction $x \in \langle B\rangle$. Thus, the claim is proven. □

(9.6) Corollary *Suppose that R is a field. Then, every G-graded R-module is free, and homogeneous bases of M, maximal free homogeneous subsets of M, and minimal homogeneous generating sets of M are the same.*

Proof. Clear from 9.5. □

(9.7) A) Let M be a G-graded R-module. By abuse of language, a G-graded sub-R-module of M is called *maximal* if it is a maximal element of the set of proper G-graded sub-R-modules of M. We denote the set of maximal G-graded sub-R-modules of M by $\text{Max}(M)$. In case $M = R$ this set coincides with the set of maximal G-graded ideals of R. If $L \subseteq M$ is a G-graded sub-R-module with $L_{[\psi]} \in \text{Max}(M_{[\psi]})$ then it holds $L \in \text{Max}(M)$.

B) The G-graded R-module M is called *simple* if $0 \in \text{Max}(M)$, and then it holds $\text{Max}(M) = \{0\}$. The G-graded ring R, considered as a G-graded R-module, is simple if and only if it is a field. On use of Zorn's Lemma it is easy to see that if M is finitely generated and $M \neq 0$ then $\text{Max}(M)$ is nonempty. If $M_{[\psi]}$ is simple then so is M. On the other hand, if M is simple then $M_{[\psi]}$ is not necessarily simple (see 9.1 C)). In particular, if $L \in \text{Max}(M)$ then it does not necessarily hold $L_{[\psi]} \in \text{Max}(M_{[\psi]})$.

C) The set $\text{Jac}(M) := \bigcap \text{Max}(M)$ is a G-graded sub-R-module of M called *the Jacobson radical of M*. By A) it follows that if M is finitely generated then $M \neq 0$ is equivalent to $\text{Jac}(M) \neq M$. The behaviour of Jacobson radicals under coarsening seems to be complicated even if $G = \mathbb{Z}$ (see [10, A.1.7]).

D) The G-graded ring R is called *local* if $\text{Card}(\text{Max}(R)) = 1$, that is, if there exists a unique maximal G-graded ideal of R, necessarily equal to $\text{Jac}(R)$. This holds if and only if the G-graded ring $R/\text{Jac}(R)$ is a field.

(9.8) Proposition *Suppose that the G-graded ring R is local. Then, the F-graded ring $R_{(\varphi)}$ is local.*

Proof. Let $\mathfrak{m}$ denote the unique maximal G-graded ideal of R. Then, the G-graded ring $R/\mathfrak{m}$ is a field, and hence the F-graded ring $R_{(\varphi)}/\mathfrak{m}_{(\varphi)} \cong (R/\mathfrak{m})_{(\varphi)}$ is a field by 9.1 B) and [13, III.1.4.6]. Therefore, $\mathfrak{m}_{(\varphi)} \in \text{Max}(R_{(\varphi)})$. Now, let $\mathfrak{n} \in \text{Max}(R_{(\varphi)})$. If $\langle\mathfrak{n}\rangle_R = R$, then there is a finite subset $E \subseteq \mathfrak{n}^{\text{hom}} \setminus 0$ and a family $(r_e)_{e\in E}$ in $R \setminus 0$ with $\sum_{e\in E} r_e e = 1$, and by taking homogeneous components of degree 0 we can suppose that r_e is homogeneous for every $e \in E$. But this implies that $\deg(r_e) = -\deg(e) \in \mathrm{Im}(\varphi)$ for every $e \in E$, hence the contradiction $1 \in \mathfrak{n}$. Thus, we have $\langle \mathfrak{n} \rangle_R \subseteq \mathfrak{m}$ and hence $\mathfrak{n} = \mathfrak{m}_{(\varphi)}$, proving that $R_{(\varphi)}$ is local. □

(9.9) Proposition *a) Let M be a G-graded R-module, and let $x \in M$. Then, it holds $x \in \mathrm{Jac}(M)$ if and only if for every simple G-graded R-module S and every morphism $p : M \to S$ in $\mathsf{GrMod}^G(R)$ it holds $p(x) = 0$.*

b) Let $x \in R$. Then, it holds $x \in \mathrm{Jac}(R)$ if and only if $x \in (0 :_R S)$ for every simple G-graded R-module S.

Proof. a) If $x \in \mathrm{Jac}(M)$, S is a simple G-graded R-module, and $p : M \to S$ is a morphism in $\mathsf{GrMod}^G(R)$ with $p \neq 0$, then simplicity of S implies that p is an epimorphism, hence $S \cong M/\mathrm{Ker}(p)$, therefore $\mathrm{Ker}(p) \in \mathrm{Max}(M)$ and thus $x \in \mathrm{Ker}(p)$. Conversely, suppose that $p(x) = 0$ for every simple G-graded R-module S and every morphism $p : M \to S$ in $\mathsf{GrMod}^G(R)$. If $N \in \mathrm{Max}(M)$ and $p : M \twoheadrightarrow M/N$ denotes the canonical epimorphism, then M/N is simple, so that $p(x) = 0$, hence $x \in N$. This shows $x \in \mathrm{Jac}(M)$.

b) If $x \in \mathrm{Jac}(R)$, S is a simple G-graded R-module, $g \in G$, and $y \in S_g$, then $S(-g)$ is a simple G-graded R-module and

$$p : R \to S(-g),\ z \mapsto zy$$

is a morphism in $\mathsf{GrMod}^G(R)$, so that $xy = p(x) = 0$ by a), and therefore $x \in (0 :_R S)$. Conversely, suppose that $x \in (0 :_R S)$ for every simple G-graded R-module S. If $\mathfrak{m} \in \mathrm{Max}(R)$, then $R/\mathfrak{m}$ is simple, so that $x \in (0 :_R R/\mathfrak{m}) = \mathfrak{m}$. This shows $x \in \mathrm{Jac}(R)$. □

(9.10) Proposition *a) If $f : M \to N$ is a morphism in $\mathsf{GrMod}^G(R)$ then it holds $f(\mathrm{Jac}(M)) \subseteq \mathrm{Jac}(N)$.*

b) If M is a G-graded R-module and $N \subseteq M$ is a G-graded sub-R-module then $(\mathrm{Jac}(M) + N)/N \subseteq \mathrm{Jac}(M/N)$.

c) If M is a G-graded R-module then $\mathrm{Jac}(R)M \subseteq \mathrm{Jac}(M)$.

Proof. a) Let $x \in \mathrm{Jac}(M)$, let S be a simple G-graded R-module, and let $p : N \to S$ be a morphism in $\mathsf{GrMod}^G(R)$. Then, it holds $p(f(x)) = 0$ and hence $f(x) \in \mathrm{Jac}(N)$ by 9.9 a).

b) As $(\mathrm{Jac}(M)+N)/N$ is the image of $\mathrm{Jac}(M)$ under the canonical epimorphism $M \twoheadrightarrow M/N$ this follows from a).

c) Let $x \in \mathrm{Jac}(R)M$, let S be a simple G-graded R-module, and let $p : M \to S$ be a morphism in $\mathsf{GrMod}^G(R)$. Then, $p(x) \in \mathrm{Jac}(R)S \subseteq (0 :_R S)S = 0$ by 9.9 b), hence $x \in \mathrm{Jac}(M)$ by 9.9 a). □

(9.11) Corollary *Let M be a G-graded R-module.*

a) Let $N \subseteq M$ be a G-graded sub-R-module. If M/N is finitely generated then $N + \mathrm{Jac}(M) = M$ implies $N = M$.

b) Let $\mathfrak{a} \subseteq R$ be a G-graded ideal with $\mathfrak{a} \subseteq \mathrm{Jac}(R)$. If $\mathfrak{a}$ is nilpotent or M is finitely generated then $\mathfrak{a}M = M$ implies $M = 0$.

c) Let $N \subseteq M$ be a G-graded sub-R-module, and let $\mathfrak{a} \subseteq R$ be a G-graded ideal with $\mathfrak{a} \subseteq \mathrm{Jac}(R)$. If $\mathfrak{a}$ is nilpotent or M/N is finitely generated then $N + \mathfrak{a}M = M$ implies $N = M$.

Proof. a) It holds $\mathrm{Jac}(M/N) \subseteq M/N = (N + \mathrm{Jac}(M))/N \subseteq \mathrm{Jac}(M/N)$ by 9.10 b), hence $\mathrm{Jac}(M/N) = M/N$, and thus $M/N = 0$ by 9.7 C).

b) If M is finitely generated, then this is clear by 9.10 c) and 9.7 C). If $\mathfrak{a}$ is nilpotent, then there is an $n \in \mathbb{N}$ with $\mathfrak{a}^n = 0$, and then $M = \mathfrak{a}M$ implies $M = \mathfrak{a}^n M = 0$.

c) If M/N is finitely generated, then this is clear by a) and 9.10 c). If $\mathfrak{a}$ is nilpotent, then this follows on use of b). □

10. Minimal resolutions

Following [4, Chapter 7] and [5, Kapitel 7] we introduce and characterise minimal projective and minimal free resolutions, by means of the notion of projective and free cover. The main result shows that over a coherent, local graded ring every finitely presented graded module has a minimal free resolution of finite rank.

Throughout this section let M be a G-graded R-module.

(10.1) A) A morphism $h : M \to N$ in $\mathsf{GrMod}^G(R)$ is called *minimal* if for every G-graded sub-R-module $L \subseteq M$ with $\mathrm{Ker}(h) + L = M$ it holds $L = M$. Clearly, h is minimal if and only if its coimage is minimal. If $(h_i)_{i\in I}$ is a family of morphisms in $\mathsf{GrMod}^G(R)$ then $\bigoplus_{i\in I} h_i$ is minimal if and only if h_i is minimal for every $i \in I$. Hence, if (C, d) is a complex in $\mathsf{GrMod}^G(R)$, then d is minimal if and only if d_n is minimal for every $n \in \mathbb{Z}$.

B) Obviously, if $h : M \to N$ is a morphism in $\mathsf{GrMod}^G(R)$ such that $h_{[\psi]}$ is minimal then so is h.

(10.2) A) A *projective cover of* M is a minimal epimorphism $p : P \twoheadrightarrow M$ in $\mathsf{GrMod}^G(R)$ such that P is projective. A *free cover of* M is a minimal epimorphism $p : P \twoheadrightarrow M$ in $\mathsf{GrMod}^G(R)$ such that P is free.

B) If $p : P \to M$ is a morphism in $\mathsf{GrMod}^G(R)$ such that $p_{[\psi]}$ is a projective (or free) cover of $M_{[\psi]}$, then p is a projective cover of M. Conversely, if $p : P \to M$ is a projective (or free) cover of M and if moreover $p_{[\psi]}$ is minimal, then $p_{[\psi]}$ is a projective (or free) cover of $M_{[\psi]}$. Indeed, this follows from [13, III.2.1.3, III.2.2.3, III.2.4.6] and 10.1 B).

(10.3) Proposition *If $p : P \twoheadrightarrow M$ and $q : Q \twoheadrightarrow M$ are projective covers of M, then there exists an isomorphism $f : P \xrightarrow{\cong} Q$ in $\mathsf{GrMod}^G(R)$ with $q \circ f = p$.*

Proof. As P is projective and q is an epimorphism there exists a morphism $f : P \to Q$ in $\mathsf{GrMod}^G(R)$ with $q \circ f = p$. As p is an epimorphism it is readily checked that $\mathrm{Ker}(q) + \mathrm{Im}(f) = Q$, hence minimality of q implies that f is an epimorphism. As Q is projective there exists a section s of f, so that $P = \mathrm{Im}(s) \oplus \mathrm{Ker}(f)$. If $L \subseteq P$ is a G-graded sub-R-module with $\mathrm{Ker}(f) + L = P$, then $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(p)$ and minimality of p imply $L = P$. Thus, f is minimal, and so we get $\mathrm{Im}(s)$. Thus, s is an isomorphism, and therefore f is an isomrphism, too. □

(10.4) Corollary *If $p : P \twoheadrightarrow M$ is a projective cover of M and if moreover M has a free cover, then p is a free cover of M.*

Proof. Clear from 10.3. □

(10.5) A) A *minimal projective resolution of* M is a projective resolution $((P,d),p)$ of M in $\mathsf{GrMod}^G(R)$ such that p_0 is minimal and that d_n is minimal for every $n \in \mathbb{N}$, that is, such that p_0 is a projective cover of M and that the coimage of d_n is a projective cover of the image of d_n for every $n \in \mathbb{N}$. A *minimal free resolution of* M is a free resolution $((P,d),p)$ of M in $\mathsf{GrMod}^G(R)$ such that p_0 is minimal and that d_n is minimal for every $n \in \mathbb{N}$, that is, such that p_0 is a free cover of M and that the coimage of d_n is a free cover of the image of d_n for every $n \in \mathbb{N}$.

B) If $((P,d),p)$ is a left resolution of M such that $((P_{[\psi]}, d_{[\psi]}), p_{[\psi]})$ is a minimal projective (or free) resolution of $M_{[\psi]}$, then $((P,d),p)$ is a minimal projective resolution of M by [13, III.2.1.3] and 10.2.

C) If $((P,d),p))$ and $((Q,e),q)$ are minimal projective resolutions of M, then it follows from 10.3 that there exists an isomorphism $h : (P,d) \to (Q,e)$ in $\mathsf{Co}(\mathsf{GrMod}^G(R))$ with $q \circ h = p$. Hence, if $((P,d),p))$ is a minimal projective resolutions of M and if moreover M has a minimal free resolution, then $((P,d),p)$ is a minimal free resolution of M.

D) Suppose that M has a minimal free resolution $((P,d),p)$, and let $n \in \mathbb{Z}$. There is a unique family $(b_g^{(n)})_{g\in G}$ of cardinal numbers such that the G-quasigraded set $(\coprod_{g\in G} b_g^{(n)}, (b_g^{(n)})_{g\in G})$ is a basis of P_n (see [13, III.1.1.1, III.2.2.2]). Since C) implies that this family does not depend on $((P,d),p)$ but only on M we denote it by $b_R^n(M) = (b_R^n(M)_g)_{g\in G}$ and call it *the n-th Betti family of M.*

(10.6) A) The G-graded ring R is called *perfect* if every G-graded R-module has a projective cover, and this is equivalent to every G-graded R-module having a minimal projective resolution. A G-graded ideal $\mathfrak{a} \subseteq R$ is called *transfinitely nilpotent* if for every sequence $(x_i)_{i\in\mathbb{N}_0}$ in $\mathfrak{a}^{\mathrm{hom}}$ there is an $n \in \mathbb{N}_0$ with $\prod_{i=0}^n a_i = 0$.

B) In the ungraded case the following statements are equivalent by [1]: R is perfect; R is the product of a finite family of local rings with transfinitely nilpotent maximal ideals; if M is an R-module, then the weak dimension[3] of M and the projective dimension of M coincide; if $n \in \mathbb{N}_0$, then the projective dimension of inductive limits of R-modules of projective dimension at most n is at most n; inductive limits of projective r-modules are projective; the ordered set of principal ideals of R, furnished with $\subseteq$, is Artinian; every set of orthogonal idempotents in R is finite, and every R-module M with $M \neq 0$ has a simple sub-R-module.

C) It would be interesting to formulate and prove G-graded versions of the characterisations in B). Several of them suggest that if $R_{[\psi]}$ is perfect, then R is so.

(10.7) Proposition *Let $h : M \to N$ be a morphism in $\mathsf{GrMod}^G(R)$, let $\mathfrak{a} \subseteq R$ be a G-graded ideal such that $\mathfrak{a} \subseteq \mathrm{Jac}(R)$, and suppose that $\mathfrak{a}$ is nilpotent or that M is finitely generated. If $\mathrm{Ker}(h) \subseteq \mathfrak{a}M$, then h is minimal.*

Proof. Let $\mathrm{Ker}(h) \subseteq \mathfrak{a}M$, and let $L \subseteq M$ be a G-graded sub-R-module with $\mathrm{Ker}(h) + L = M$. Then, it holds $\mathfrak{a}M + L = M$, and so 9.11 c) implies $L = M$. Thus, h is minimal. □

(10.8) Corollary *Let $h : M \to N$ be a morphism in $\mathsf{GrMod}^G(R)$, let $\mathfrak{a} \subseteq R_{[\psi]}$ be an H-graded ideal such that $\mathfrak{a} \subseteq \mathrm{Jac}(R_{[\psi]})$, and suppose that $\mathfrak{a}$ is nilpotent or that M is finitely generated. If $\mathrm{Ker}(h_{[\psi]}) \subseteq \mathfrak{a}M_{[\psi]}$, then h is minimal.*

[3] that is, the infimum of the lengths of all flat resolutions of M

Proof. Clear by 10.7 and 10.1 B). □

(10.9) Lemma *Let $E \subseteq M^{\mathrm{hom}}$, and let $p : M \twoheadrightarrow M/\operatorname{Jac}(R)M$ denote the canonical projection.*

a) If $(e_i)_{i\in I}$ is a homogeneous generating family of M then $(p(e_i))_{i\in I}$ is a homogeneous generating family of $M/\operatorname{Jac}(R)M$, and if $(p(e_i))_{i\in I}$ is a minimal homogeneous generating family of $M/\operatorname{Jac}(R)M$ then $(e_i)_{i\in I}$ is a minimal homogeneous generating family of M.

b) Suppose that $\operatorname{Jac}(R)$ is nilpotent or that M is finitely generated. If $(p(e_i))_{i\in I}$ is a homogeneous generating family of $M/\operatorname{Jac}(R)M$ then $(e_i)_{i\in I}$ is a homogeneous generating family of M, and if $(e_i)_{i\in I}$ is a minimal homogeneous generating family of M then $(p(e_i))_{i\in I}$ is a minimal homogeneous generating family of $M/\operatorname{Jac}(R)M$.

Proof. a) The first statement is obvious. Suppose that $(p(e_i))_{i\in I}$ is a homogeneous generating family of $M/\operatorname{Jac}(R)M$ and that $(e_i)_{i\in I}$ is not a minimal homogeneous generating family of M. Then, by a) there is an $i \in I$ with $e_i \in \langle e_j \mid j \in I \setminus \{i\}\rangle_R$, implying $p(e_i) \in \langle p(e_j) \mid j \in I \setminus \{i\}\rangle_{R/\operatorname{Jac}(R)}$, and hence $(p(e_i))_{i\in I}$ is not minimal.

b) If

$$\langle p(e_i) \mid i \in I\rangle_{R/\operatorname{Jac}(R)} = M/\operatorname{Jac}(R)M$$

then it holds

$$M = \langle e_i \mid i \in I\rangle_R + \operatorname{Jac}(R)M,$$

hence $M = \langle e_i \mid i \in I\rangle_R$ by 9.11 c). Suppose that $(e_i)_{i\in I}$ is a homogeneous generating family of M and that $(p(e_i))_{i\in I}$ is not a minimal homogeneous generating family of $M/\operatorname{Jac}(R)M$. Then, by a) there is a proper subset $J \subsetneqq I$ such that $(p(e_i))_{i\in J}$ is a homogeneous generating family of $M/\operatorname{Jac}(R)M$, and b) implies that $(p(e_i))_{i\in J}$ is a homogeneous generating family of M, hence $(e_i)_{i\in I}$ is not minimal. □

(10.10) Lemma *Let $h : M \twoheadrightarrow N$ be an epimorphism in $\mathsf{GrMod}^G(R)$, and let $j : \operatorname{Ker}(h) \hookrightarrow M$ denote the canonical injection. Then, the following statements are equivalent:*

(i) It holds $\operatorname{Ker}(h) \subseteq \operatorname{Jac}(R)M$;

(ii) It holds $(R/\operatorname{Jac}(R)) \otimes_R j = 0$;

(iii) $(R/\operatorname{Jac}(R)) \otimes_R h$ is a monomorphism;

(iv) $(R/\operatorname{Jac}(R)) \otimes_R h$ is an isomorphism.

Proof. If (i), then for $x \in \operatorname{Ker}(h)$ it holds $j(x) \in \operatorname{Jac}(R)M$, hence

$$((R/\operatorname{Jac}(R)) \otimes_R j)(x + \operatorname{Jac}(R)M) = 0,$$

and thus (ii). If (ii), then

$$\operatorname{Ker}((R/\operatorname{Jac}(R)) \otimes_R h) = \operatorname{Im}((R/\operatorname{Jac}(R)) \otimes_R j) = 0,$$

and hence (iii). If (iii), then rightexactness of $(R/\operatorname{Jac}(R)) \otimes_R \bullet$ implies (iv). If (iv), then on use of the canonical identification of $(R/\operatorname{Jac}(R)) \otimes_R M$ and $M/\operatorname{Jac}(R)M$ we see that

$$x + \operatorname{Jac}(R)M \in \operatorname{Ker}((R/\operatorname{Jac}(R)) \otimes_R h) = 0$$

for $x \in \operatorname{Ker}(h)$, hence (i). □

(10.11) Proposition *Let $h : M \twoheadrightarrow N$ be an epimorphism in $\mathsf{GrMod}^G(R)$. We consider the following statements:*

(i) $\mathrm{Ker}(h) \subseteq \mathrm{Jac}(R)M$.

(ii) The morphism

$$(R/\mathrm{Jac}(R)) \otimes_R h : M/\mathrm{Jac}(R)M \to N/\mathrm{Jac}(R)N$$

in $\mathsf{GrMod}^G(R/\mathrm{Jac}(R))$ is an isomorphism;

(iii) If $(e_i)_{i \in I}$ is a minimal homogeneous generating family of M then $(h(e_i))_{i \in I}$ is a minimal homogeneous generating family of N;

(iv) There is a minimal homogeneous generating family $(e_i)_{i \in I}$ of M such that $(h(e_i))_{i \in I}$ is a minimal homogeneous generating family of N;

a) It holds (i)⇔(ii)⇒(iii), and if M has a minimal homogeneous generating family then it holds (iii)⇒(iv).

b) Suppose that R is local. If M is finitely generated, or if $\mathrm{Jac}(R)$ is nilpotent and M has a minimal homogeneous generating family, then it holds

(i) ⇔ (ii) ⇔ (iii) ⇔ (iv).

Proof. a) The equivalence of (i) and (ii) holds by 10.10. On use of 10.9 it is seen that (ii) implies (iii). If M has a minimal homogeneous generating family, then (iii) obviously implies (iv).

b) Suppose that (iv) holds. Then, 10.9 and 9.6 imply (ii), and thus the claim is proven. □

(10.12) Proposition *Suppose that R is local, and suppose that M is finitely generated, or that $\mathrm{Jac}(R)$ is nilpotent and that M has a minimal homogeneous generating family. Then, M has a free cover.*

Proof. Let $(e_i)_{i \in I}$ be a minimal homogeneous generating family of M, and let $g_i := \deg(e_i) \in G$ for $i \in I$. Then, $F := \bigoplus_{i \in I} R(-g_i)$ is a free G-graded R-module, and there is an epimorphism $p : F \twoheadrightarrow M$ mapping the canonical basis of F onto $(e_i)_{i \in I}$. Therefore, p is minimal by 10.11 and 10.7, hence a free cover of M. □

(10.13) A) A G-graded R-module M is called *coherent* if it is finitely generated and if moreover every finitely generated G-graded sub-R-module of M is of finite presentation. It is easy to see that if two G-graded R-modules occuring in a short exact sequence are coherent, then so is the third. Consequently, a finite direct sum of G-graded R-modules is coherent if and only if every summand is coherent (see [2, X.3 Exercises 10–11] and [8]). If $M_{[\psi]}$ is coherent, then so is M.

B) The G-graded ring R is called *coherent* if it is so considered as a G-graded R-module. From A) it follows that R is coherent if and only if every G-graded R-module of finite presentation is coherent. Moreover, it can be shown that R is coherent if and only if products of flat G-graded R-modules are flat, and this holds if and only if $(\mathfrak{a} :_R x)$ is finitely generated for every G-graded ideal $\mathfrak{a} \subseteq R$ and every $x \in R^{\mathrm{hom}}$ (see [8, 2.3.2] for proofs in the ungraded case). If $R_{[\psi]}$ is coherent, then so is R.

C) If R is Noetherian, then it is obviously coherent. If R is absolutely flat[4], then it is coherent by B). If R is Noetherian, then polynomial algebras over R are

[4]that is, every G-graded R-module is flat

coherent, but not necessarily Noetherian (see [8, 2.3.3] for a proof in the ungraded case).

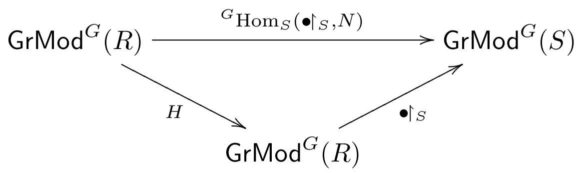
(10.14) Theorem *If R is local and coherent, then every finitely presented G-graded R-module has a minimal free resolution of finite rank.*

Proof. If every finitely presented G-graded R-module M has a free cover $p : F \twoheadrightarrow M$ such that $\mathrm{Ker}(p)$ is finitely presented, then it follows by recursion that it has a minimal free resolution. Therefore, if R is coherent it suffices to show that every finitely presented G-graded R-module has a free cover with a finitely generated kernel. But this follows from 10.12 and [2, X.1.4 Proposition 6]. □

11. Duality functors

Along the lines of [4, Chapter 9] we start building a theory of duality in the context of arbitrarily graded modules. The key idea is to replace the functors $\bullet_0$ of taking components of degree 0 by restriction functors $\bullet_{(\varphi)}$.

(11.1) A) Let $S \subseteq R$ be a G-graded subring, and let N be a G-graded S-module. Then, there is a canonical contravariant functor $H : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R)$ such that the diagram



of categories commutes[5], where for a G-graded R-module M the structure of G-graded R-module of $H(M)$ on ${}^G\mathrm{Hom}_S(M\restriction_S, N)$ is given by $(ru)(x) = u(rx)$ for $r \in R$, $u \in {}^G\mathrm{Hom}_S(M\restriction_S, N)$ and $x \in M$. If no confusion can arise, then we denote H again by ${}^G\mathrm{Hom}_S(\bullet\restriction_S, N)$.

B) Clearly, the contravariant functor

$$ {}^G\mathrm{Hom}_S(\bullet\restriction_S, N) : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R) $$

is leftexact, and it is exact if and only if the contravariant functor

$$ {}^G\mathrm{Hom}_S(\bullet\restriction_S, N) : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(S) $$

is exact.

C) If N is injective in $\mathsf{GrMod}^G(S)$ then the contravariant functor

$$ {}^G\mathrm{Hom}_S(\bullet\restriction_S, N) : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R) $$

is exact by [13, III.2.4.9] and B).

(11.2) A) Let $g \in G$. Then, there are canonical isomorphisms

$$ {}^G\mathrm{Hom}_R(\bullet, \blacksquare)(g) \cong {}^G\mathrm{Hom}_R(\bullet, \blacksquare(g)) \cong {}^G\mathrm{Hom}_R(\bullet(-g), \blacksquare) $$

of contra-covariant bifunctors from $\mathsf{GrMod}^G(R) \times \mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^G(R)$.

B) If $f \in F$, then the functors $(\bullet_{(\varphi)})(f)$ and $\bullet(\varphi(f))_{(\varphi)}$ from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^F(R_{(\varphi)})$ are equal.

[5] Is H uniquely determined by this diagram?

C) Let S be an F-graded ring. If $f \in F$, then the functors $\bullet(f)^{(\varphi)}$ and $\bullet^{(\varphi)}(\varphi(f))$ from $\mathsf{GrMod}^F(S)$ to $\mathsf{GrMod}^G(S^{(\varphi)})$ are equal.

(11.3) A) Let M and N be G-graded R-modules. By 11.2 B) there is a morphism

$$\mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M, N(\varphi(f))) \to \mathrm{Hom}_{\mathsf{GrMod}^F(R_{(\varphi)})}(M_{(\varphi)}, N_{(\varphi)}(f)),\ u \mapsto u_{(\varphi)}$$

in Ab for every $f \in F$. Taking direct sums yields a morphism

$$l_\varphi(M,N) : {}^G\mathrm{Hom}_R(M,N)_{(\varphi)} \to {}^F\mathrm{Hom}_{R_{(\varphi)}}(M_{(\varphi)}, N_{(\varphi)})$$

in $\mathsf{GrMod}^F(R_F)$. This is clearly natural in M and N, and hence it gives rise to a morphism

$$l_\varphi : {}^G\mathrm{Hom}_R(\bullet, \blacksquare)_{(\varphi)} \to {}^F\mathrm{Hom}_{R_{(\varphi)}}(\bullet_{(\varphi)}, \blacksquare_{(\varphi)})$$

of contra-covariant bifunctors from $\mathsf{GrMod}^G(R) \times \mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^F(R_{(\varphi)})$.

B) The canonical morphism l_φ is not necessarily a monomorphism. Indeed, let $\varphi : 0 \to \mathbb{Z}/2\mathbb{Z}$, let $R = R_0 = \mathbb{Z}$, let $M = \mathbb{Z} \oplus \mathbb{Z}$ with $\deg(1,0) = 0$ and $\deg(0,1) = 1$, and let $u : M \to M,\ (x,y) \mapsto (x, 2y)$. Then, it holds $u \in \mathrm{Hom}_{\mathsf{GrMod}^{\mathbb{Z}/2\mathbb{Z}}(\mathbb{Z})}(M,M)$ with $u \neq \mathrm{Id}_M$, but $l_\varphi(u) = l_\varphi(\mathrm{Id}_M)$, so that $l_\varphi(M,M)$ is not injective.[6].

(11.4) Proposition *Let N be a G-graded R-module. If* $\mathrm{degsupp}(N) \subseteq F$ *then the canonical morphism* $l_\varphi(\bullet, N) : {}^G\mathrm{Hom}_R(\bullet, N)_{(\varphi)} \to {}^F\mathrm{Hom}_{R_{(\varphi)}}(\bullet_{(\varphi)}, N_{(\varphi)})$ *is a monomorphism, and if in addition* $\mathrm{degsupp}(R) \subseteq F$ *then it is an isomorphism.*

Proof. Let M be a G-graded R-module, and suppose that $\mathrm{degsupp}(N) \subseteq F$. Let $f \in F$ and let $u \in \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M, N(\varphi(f)))$ with $u_{(\varphi)} = 0$. For $g \in \mathrm{Im}(\varphi)$ it holds $u_g = 0$, and for $g \in G \setminus \mathrm{Im}(\varphi)$ it holds $u_g(M_g) \subseteq N(\varphi(f))_g = N_{\varphi(f)+g} = 0$, hence $u_g = 0$. This shows $u = 0$, and therefore $l_\varphi(M,N)$ is a monomorphism.

Now, suppose that in addition $\mathrm{degsupp}(R) \subseteq F$. Let $f \in F$ and let $u \in \mathrm{Hom}_{\mathsf{GrMod}^F(R_{(\varphi)})}(M_{(\varphi)}, N_{(\varphi)}(f))$. For $h \in F$ we set $v_{\varphi(h)} := u_h : M_h \to N(\varphi(f))_h$, and for $g \in G \setminus \mathrm{Im}(\varphi)$ we set $v_g := 0 : M_g \to N(\varphi(f))_g$. It is readily checked that $v := \bigoplus_{g \in G} v_g \in \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M, N(\varphi(f)))$ and $v_{(\varphi)} = u$. Thus, $l_\varphi(M,N)$ is an epimorphism, hence an isomorphism. □

(11.5) A) A G-graded R-module N is a cogenerator[7] in $\mathsf{GrMod}^G(R)$ if and only if for every G-graded R-module M and every $x \in M^{\mathrm{hom}} \setminus 0$ there is a $u \in \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M,N)$ with $u(x) \neq 0$.

B) As $\bullet_g = \bullet(g)_0$ for every $g \in G$ it is seen that a G-graded R-module N is a cogenerator in $\mathsf{GrMod}^G(R)$ if and only if for every G-graded R-module M and every $x \in M_0 \setminus 0$ there is a $u \in \mathrm{Hom}_{\mathsf{GrMod}^G(R)}(M,N)$ with $u(x) \neq 0$.

C) If N is a cogenerator in $\mathsf{GrMod}^G(R)$ then for every G-graded R-module M and every $x \in M^{\mathrm{hom}} \setminus 0$ there is a $u \in {}^G\mathrm{Hom}_R(M,N)$ with $u(x) \neq 0$.[8]

(11.6) Proposition *Let S be an F-graded ring, and let N be an F-graded S-module. Then, N is a cogenerator in* $\mathsf{GrMod}^F(S)$ *if and only if $N^{(\varphi)}$ is a cogenerator in* $\mathsf{GrMod}^G(S^{(\varphi)})$.

---

[6]Probably, l_φ is not necessarily an epimorphism – counterexample?

[7]If C is a category, then a *cogenerator in* C is a $C \in \mathrm{Ob}(\mathsf{C})$ such that the contravariant functor $\mathrm{Hom}_{\mathsf{C}}(\bullet, C) : \mathsf{C} \to \mathsf{Ens}$ is faithful.

[8]Does the converse hold?

Proof. First, suppose that N is a cogenerator in $\mathsf{GrMod}^F(S)$. Let M be a G-graded $S^{(\varphi)}$-module, and let $x \in M_0 \setminus 0$, hence $x \in M_{(\varphi)}^{\mathrm{hom}} \setminus 0$. Then, there is a $u \in \mathrm{Hom}_{\mathsf{GrMod}^F(S)}(M_{(\varphi)}, N)$ with $u(x) \neq 0$, and by 11.4 we can consider u as an element of $\mathrm{Hom}_{\mathsf{GrMod}^G(S^{(\varphi)})}(M, N^{(\varphi)})$. Thus, $N^{(\varphi)}$ is a cogenerator in $\mathsf{GrMod}^G(S^{(\varphi)})$.

Conversely, suppose that $N^{(\varphi)}$ is a cogenerator in $\mathsf{GrMod}^G(S^{(\varphi)})$. Let M be an F-graded S-module, and let $x \in M_0 \setminus 0$, hence $x \in (M^{(\varphi)})^{\mathrm{hom}} \setminus 0$. Then, there is a $u \in \mathrm{Hom}_{\mathsf{GrMod}^G(S^{(\varphi)})}(M^{(\varphi)}, N^{(\varphi)})$ with $u(x) \neq 0$, and it holds $u_{(\varphi)} \in \mathrm{Hom}_{\mathsf{GrMod}^F(S)}(M, N)$ with $u(x) \neq 0$. Thus, N is a cogenerator in $\mathsf{GrMod}^F(S)$. □

For the rest of this section, in order to increase readability but by abuse of language we write $R_F := R_{(\varphi)}$ and $R_F^G := (R_{(\varphi)})^{(\varphi)}$, as well as $\bullet_F := \bullet_{(\varphi)} : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^F(R_F)$ and $\bullet_F^G := (\bullet_{(\varphi)})^{(\varphi)} : \mathsf{GrMod}^F(R_F) \to \mathsf{GrMod}^G(R_F^G)$. Moreover, we denote the functor of scalar restriction by means of the canonical injection $R_F^G \hookrightarrow R$ just by $\upharpoonright: \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R_F^G)$. Furthermore, let E be an F-graded R_F-module.

(11.7) The contravariant functor

$$^G\mathrm{Hom}_{R_F^G}(\bullet\upharpoonright, E^G) : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R)$$

is denoted by $D_{\varphi,E}$ and called *the (φ, E)-duality functor on* $\mathsf{GrMod}^G(R)$. If no confusion can arise we write D_φ, D_E or D instead of $D_{\varphi,E}$.

(11.8) Proposition *The contravariant functor $D_{\varphi,E}$ is leftexact, and if E is injective in $\mathsf{GrMod}^F(R_F)$ then it is exact.*

Proof. a) is clear by 11.1. b) As $\bullet^G$ is a left adjoint of the exact functor $\bullet_F$, it preserves injective objects by [12, Theorem 3.2.8]. Therefore, E^G is injective in $\mathsf{GrMod}^G(R_F^G)$, and then the claim follows from 11.1. □

(11.9) Let $g \in G$. Then, by 11.2 A) there is a canonical isomorphism

$$D_{\varphi,E}(\bullet)(g) \cong D_{\varphi,E}(\bullet(-g))$$

of contravariant functors from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^G(R)$.

(11.10) Proposition *The canonical morphism*

$$l_\varphi(\bullet, E^G) : D_{\varphi,E}(\bullet)_F \to {}^F\mathrm{Hom}_{R_F}(\bullet_F, E)$$

of contravariant functors from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^F(R_F)$ is an isomorphism.

Proof. Clear from 11.4. □

(11.11) Corollary *There is a canonical isomorphism*

$$D_{\varphi,E}(D_{\varphi,E}(\bullet))_F \overset{\cong}{\to} {}^F\mathrm{Hom}_{R_F}({}^F\mathrm{Hom}_{R_F}(\bullet_F, E), E)$$

of functors from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^F(R_F)$.

Proof. We write $D = D_{\varphi,E}$. By 11.10 there is a canonical isomorphism

$$l : D(D(\bullet))_F \overset{\cong}{\to} {}^F\mathrm{Hom}_{R_F}(D(\bullet)_F, E)$$

and hence a canonical isomorphism

$${}^F\mathrm{Hom}_{R_F}(l, E) : {}^F\mathrm{Hom}_{R_F}({}^F\mathrm{Hom}_{R_F}(D(\bullet)_F, E), E) \overset{\cong}{\to} {}^F\mathrm{Hom}_{R_F}(D(D(\bullet))_F, E),$$

and then ${}^F\mathrm{Hom}_{R_F}(l, E)^{-1} \circ l$ fulfils the claim. □

(11.12) If M is a G-graded R-module, then there is a canonical morphism

$$\gamma_{\varphi,E}(M) : M \to D_{\varphi,E}(D_{\varphi,E}(M))$$

in $\mathsf{GrMod}^G(R)$ with $\gamma_{\varphi,E}(M)(x)(u) = u(x)$ for $x \in M$ and $u \in D_{\varphi,E}(M)$. These morphisms being clearly natural in M we end up with a canonical morphism

$$\gamma_{\varphi,E} : \mathrm{Id}_{\mathsf{GrMod}^G(R)} \to D_{\varphi,E} \circ D_{\varphi,E}$$

of functors from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^G(R)$.

(11.13) Proposition *If E is a cogenerator in $\mathsf{GrMod}^F(R_F)$, then the canonical morphism $\gamma_{\varphi,E} : \mathrm{Id}_{\mathsf{GrMod}^G(R)} \to D_{\varphi,E} \circ D_{\varphi,E}$ of functors from $\mathsf{GrMod}^G(R)$ to $\mathsf{GrMod}^G(R)$ is a monomorphism.*

Proof. It follows from 11.6 that E^G is a cogenerator in $\mathsf{GrMod}^G(R_F^G)$, and then the claim follows immediately. □

(11.14) A) The contravariant functors

$${}^F\mathrm{Hom}_{R_F}(\bullet, E) : \mathsf{GrMod}^F(R_F) \to \mathsf{GrMod}^F(R_F)$$

and

$$D_{\mathrm{Id}_F,E} : \mathsf{GrMod}^F(R_F) \to \mathsf{GrMod}^F(R_F)$$

are equal. We denote by $\mathbb{F}_{\varphi,E}$ the full subcategory of $\mathsf{GrMod}^G(R)$ whose objects are the G-graded R-modules M such that for every $g \in G$ the canonical morphism

$$\gamma_{\mathrm{Id}_F,E}(M(g)_F) : M(g)_F \to {}^F\mathrm{Hom}_{R_F}({}^F\mathrm{Hom}_{R_F}(M(g)_F, E), E)$$

of F-graded R_F-modules is an epimorphism.

B) Clearly, if $g \in G$ then $\bullet(g) : \mathsf{GrMod}^G(R) \to \mathsf{GrMod}^G(R)$ induces by restriction and coastriction a functor $\bullet(g) : \mathbb{F}_{\varphi,E} \to \mathbb{F}_{\varphi,E}$.

(11.15) Proposition *Let M be a G-graded R-module. It holds $M \in \mathrm{Ob}(\mathbb{F}_{\varphi,E})$ if and only if $\gamma_{\varphi,E}(M)$ is an epimorphism.*

Proof. Clearly, $\gamma_{\varphi,E}(M)$ is an epimorphism if and only if $\gamma_{\varphi,E}(M)_g$ is an epimorphism for every $g \in G$. But since $\bullet_g = \bullet(g)_0$ for every $g \in G$, this holds if and only if $\gamma_{\varphi,E}(M(g))_F$ is an epimorphism for every $g \in G$. On use of 11.11 this implies the claim. □

References

[1] H. BASS, *Finitistic dimension and a homological generalization of semi-primary rings.* Trans. Amer. Math. Soc. 95 (1960), 466–488.

[2] N. BOURBAKI, *Éléments de mathématique: Algèbre.* Springer-Verlag, Berlin, 2006.

[3] N. BOURBAKI, *Éléments de mathématique: Algèbre commutative.* Springer-Verlag, Berlin, 2006.

[4] M. P. BRODMANN, *Around Castelnuovo-Mumford regularity.* Lecture notes, Zürich, 2010.

[5] M. P. BRODMANN, R. BOLDINI, F. ROHRER, *Kommutative Algebra.* Lecture notes, Zürich, 2010.

[6] M. P. BRODMANN, S. FUMASOLI, F. ROHRER, *First lectures on local cohomology.* Lecture notes, Zürich, 2007.

[7] M. P. BRODMANN, R. Y. SHARP, *Local cohomology.* Cambridge studies in advanced mathematics, Cambridge University Press, Cambridge, 1998.

[8] S. GLAZ, *Commutative coherent rings.* Lecture Notes in Mathematics 1371, Springer-Verlag, Berlin, 1989.

[9] A. GROTHENDIECK, *Sur quelques points d'algèbre homologique.* Tôhoku Math. J. (2) 9 (1957), 119–221.

[10] C. NĂSTĂSESCU, F. VAN OYSTAEYEN, *Graded ring theory.* North-Holland Mathematical Library 28, North-Holland Publishing Co., Amsterdam, 1982.

[11] C. NĂSTĂSESCU, F. VAN OYSTAEYEN, *Methods of graded rings.* Lecture Notes in Mathematics 1836, Springer-Verlag, Berlin, 2004.

[12] N. POPESCU, *Abelian categories with applications to rings and modules.* Academic Press, London, 1973.

[13] F. ROHRER, *Toric schemes.* Dissertation, Zürich, 2010.

[14] J. J. ROTMAN, *An introduction to homological algebra.* Pure and Applied Mathematics 85, Academic Press, New York, 1979.

[15] S. YASSEMI, *Coassociated primes of modules over a commutative ring.* Math. Scand. 80 (1997), 175–187.